Pages 2-3, Assigned Tuesday 9/21/10

Consider non-empty sets A, B and a function $f: A \to B$.
Also, consider the subsets $A_1, A_2 \subset A$ and $B_1, B_2 \subset B$.
Use the definitions $f(A_1) = \{f(a) \in B: a \in A_1\}$ and $f^{-1}(B_1) = \{a \in A: f(a) \in B_1\}$.

1. Show that:

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
(b) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. Show an example for $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.
(c) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
(d) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
(e) $f^{-1}(B \setminus B_1) = A \setminus f^{-1}(B_1)$.

Proof:

(a) We know that $b \in f(A_1 \cup A_2)$
$\Leftrightarrow a \in A_1 \cup A_2$ where $f(a) = b$
$\Leftrightarrow a \in A_1$ or $a \in A_2$ where $f(a) = b$
$\Leftrightarrow b \in f(A_1)$ or $b \in f(A_2)$.
$\Leftrightarrow b \in f(A_1) \cup f(A_2)$.
$\therefore f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(b) We have that $b \in f(A_1 \cap A_2)$
$\Rightarrow \exists\, a \in A_1 \cap A_2$ such that $f(a) = b$
$\Rightarrow a \in A_1$ and $a \in A_2$
$\Rightarrow b \in f(A_1)$ and $b \in f(A_2)$
$\Rightarrow b \in f(A_1) \cap f(A_2)$.
$\therefore f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

Example to show $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$. :

$f(x) = x^2$, $A_1 = \{-2\}$, $A_2 = \{2\}$.
Then $f(A_1 \cap A_2) = \emptyset \neq \{4\} = f(A_1) \cap f(A_2)$.

(c) We know that $a \in f^{-1}(B_1 \cup B_2)$
$\Leftrightarrow f(a) \in B_1 \cup B_2$
$\Leftrightarrow f(a) \in B_1$ or $f(a) \in B_2$
$\Leftrightarrow a \in f^{-1}(B_1)$ or $a \in f^{-1}(B_2)$
$\Leftrightarrow a \in f^{-1}(B_1) \cup f^{-1}(B_2)$.
$\therefore f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(d) We know that $a \in f^{-1}(B_1 \cap B_2)$
$\Leftrightarrow f(a) \in B_1 \cap B_2$
$\Leftrightarrow f(a) \in B_1$ and $f(a) \in B_2$
$\Leftrightarrow a \in f^{-1}(B_1)$ and $a \in f^{-1}(B_2)$
$\Leftrightarrow a \in f^{-1}(B_1) \cap f^{-1}(B_2)$.
$\therefore f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(e) We have that $a \in f^{-1}(B \setminus B_1)$
$\Leftrightarrow f(a) \in B \setminus B_1$
$\Leftrightarrow f(a) \in B$ and $f(a) \notin B_1$
$\Leftrightarrow a \in f^{-1}(B)$and $a \notin f^{-1}(B_1)$
$\Leftrightarrow a \in (f^{-1}(B)) \setminus f^{-1}(B_1) = A \setminus f^{-1}(B_1)$.
$\therefore f^{-1}(B \setminus B_1) = A \setminus f^{-1}(B_1)$.

2. This exercise regards the 1-1 functions:
(a) Show that if f is 1-1, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.
(b) If $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, for all $A_1, A_2 \subset A$, then f is 1-1.
(c) f is 1-1 $\Leftrightarrow f^{-1}(f(A_1)) = A_1$ for every $A_1 \subset A$.
(d) From the proof of (c), conclude that $A_1 \subset f^{-1}(f(A_1))$ for any function f.
(e) Show that if $f: A \to B$ and $g: B \to C$ are 1-1, then $g \circ f: A \to C$ is 1-1.

Proof:
(a) Assume f is 1-1. By 1(b) we have $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, so we only need to show $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$.
We have that $b \in f(A_1) \cap f(A_2)$
$\Rightarrow b \in f(A_1)$ and $b \in f(A_2)$
$\Rightarrow \exists\, a_1 \in A_1$ and $\exists\, a_2 \in A_2$ such that $f(a_1) = b = f(a_2)$.
Since f is 1–1, then $a_1 = a_2$.
Thus, $a_1 \in A_1 \cap A_2$, hence $b \in f(A_1 \cap A_2)$.
$\therefore f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

(b) Assume $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ for all $A_1, A_2 \subset A$.
Let $a, b \in A$ such that $f(a) = f(b)$.
Let $A_1 = \{a\}$ and let $A_2 = \{b\}$.
Then $f(a) = f(b) \in f(A_1) \cap f(A_2)$. Note that $f(A_1) \cap f(A_2) \neq \emptyset$.
Since $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, $f(a) = f(b) \in f(A_1 \cap A_2)$.
If a ≠ b, then $A_1 \cap A_2 = \emptyset$, hence $f(A_1 \cap A_2)$, a contradiction.
$\therefore$ $a = b$, hence f is 1-1.

(c) $\Rightarrow$: Assume f is 1-1. Let $A_1 \subset A$.
We have that $a \in f^{-1}(f(A_1)) \Rightarrow f(a) \in f(A_1) \Rightarrow \exists\, a_1 \in A_1$ such that $f(a_1) = f(a)$.
Since f is 1-1, then $a = a_1$, hence $a \in A_1$.
And $a \in A_1 \Rightarrow f(a) \in f(A_1) \Rightarrow a \in f^{-1}(f(A_1))$.
$\therefore$ $f^{-1}(f(A_1)) = A_1$.

$\Leftarrow$: Assume $f^{-1}(f(A_1)) = A_1$ for every $A_1 \subset A$.
Let $a, b \in A$ such that $f(a) = f(b)$. Let $A_1 = \{a\}$.
Then $f(a) \in f(A_1)$, hence $f(b) \in f(A_1)$. Thus $b \in f^{-1}(f(A_1))$.
Since $f^{-1}(f(A_1)) = A_1$, then $b \in A_1$, hence $a = b$.
$\therefore$ f is 1-1.

(d) Let $A_1 \subset A$. Let $a \in A_1$. Let f be a function $f: A \to B$.
Then $f(a) \in f(A_1)$, hence $a \in f^{-1}(f(A_1))$. $\therefore$ $A_1 \subset f^{-1}(f(A_1))$.

(e) Let $x, y \in A$ such that $(g \circ f)(x) = (g \circ f)(y)$.
Then $g(f(x)) = g(f(y))$.
Since g is 1-1, we have that $f(x) = f(y)$.
Since f is 1-1, we have that $x = y$.
$\therefore g \circ f: A \to C$ is 1-1.

3. This exercise regards the onto functions.
(a) Show that $f(f^{-1}(B_1)) \subset B_1$ for any function.
(b) f is onto $\Leftrightarrow f(f^{-1}(B_1)) = B_1$ for all $B_1 \subset B$.
(c) Show that if $f: A \to B$ and $g: B \to C$ are onto functions, then $g \circ f: A \to C$ is 1-1 is onto.

Proof:
(a) Let f be a function $f: A \to B$. Let $b \in f(f^{-1}(B_1))$. Then $f^{-1}(b) \in f^{-1}(B_1)$.
Thus $b \in B_1$. $\therefore$ $f(f^{-1}(B_1)) \subset B_1$.

(b) $\Rightarrow$: Assume f is onto.
We have by part (a) that $f^{-1}(B_1)) \subset B_1$, so we only need to show that $B_1 \subset f(f^{-1}(B_1))$.
Let $b \in B_1$. Since f is onto, then $\exists\ a \in A$ such that $f(a) = b$. Thus $f^{-1}(b) = a$.
So then $b = f(a) = f(f^{-1}(b)) \in f(f^{-1}(B_1))$. $\therefore$ $B_1 \subset f(f^{-1}(B_1))$.
$\Leftarrow$: Assume $f(f^{-1}(B_1)) = B_1$ for all $B_1 \subset B$.
Let $b \in B$. Let $B_1 = \{b\}$. Then $f(f^{-1}(\{b\})) = \{b\}$ by our assumption.
Thus $f(f^{-1}(b)) = b$. $\therefore$ f is onto.

(c) Let $f: A \to B$, $g: B \to C$ be onto fuctions.
Let $z \in C$ then since g is onto, $\exists\ w \in B$ such that $g(w) = z$.
And since f is onto, $\exists\ x \in A$ such that $f(x) = w$.
So then $(g \circ f)\ (x) = z$, hence $g \circ f$ is onto.

4. This exercise regards the 1-1 and onto functions.
(a) Show that if $A \to B$ and $g: B \to C$ are 1-1 and onto functions, then $g \circ f: A \to C$ is 1-1 and onto, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof:
(a) By 2(a), $g \circ f: A \to C$ is onto. And by 3(c) $g \circ f: A \to C$ is 1-1.
$(g \circ f)^{-1}(C) = \{a \in A |\ (g \circ f)(a) \in C\}$.
Let $a \in (g \circ f)^{-1}(\mathrm{C})$. Then $(g \circ f)(a) = c$ for some $c \in C$.
So then $(g(f(a)) = c$. Thus $f(a) = g^{-1}(c)$, hence $a = (f^{-1} \circ g^{-1})(c)$.
$\therefore$ $a \in f^{-1} \circ g^{-1}(C)$.

5. Think about the polynomial, rational, exponential, logarithmic, trigonometric functions. Examine their 1-1 and onto properties.
Polynomial Functions: Polynomials of even degree are not 1-1 or onto.
Polynomials of odd degree that are factorable over the reals are not 1-1, but are onto.
Exponential functions: 1-1, but not onto.
Logarithmic functions: 1-1, but the domain is not defined on all of $\mathbb{R}$.
Trigonometric functions:
Sine: Not 1-1 and not onto.
Cosine: Not 1-1 and not onto.
Tangent: 1-1 and onto, but not defined on all of $\mathbb{R}$.