Pages 4, 5, Assigned Friday, 9/24/10

Use Definition 2.3 to define $A \sim B \Leftrightarrow \exists f: A \to B$ that is 1-1 and onto.
A and B have the same cardinal number if $A \sim B$.
Use the notation card(A) for the cardinal number of A.
card(A) can be identified with the equivalence class of the set A: $[A] = \{\text{sets } B : A \sim B\}$.

Examples: card$\{1, 2, ..., n\} = n$, card($\mathbb{N}$) = card($\mathbb{Z}$) = card($\mathbb{Q}$) = $\aleph_0$, card($\mathbb{R}$) = $\mathfrak{c}$.

1. Define an order in the set of cardinal numbers:
card(A) ≤ card(B) $\Leftrightarrow \exists f: A \to B$ that is 1-1.

(a) Show that card(A) ≤ card(B) $\Leftrightarrow \exists g: B \to A$ that is onto.

Proof:
$\Rightarrow$: Assume card(A) ≤ card(B). Then $\exists f: A \to B$ that is 1-1.

Let $x \in A$. Since f is 1-1, then by 2(c), $f^{-1}(f(A)) = A$.

Thus, $\forall\, b \in f(A), f^{-1}(b) \in A$. Thus, we can define the following:

Define $g: B \to A$ by $g(b) = \begin{cases} f^{-1}(b) \text{ if } b \in f(A) \\ x \text{ otherwise} \end{cases}$

Let $a \in A$. Then $f(a) \in f(A) \subset B$. Thus, $\exists\, b \in B$ such that $f^{-1}(b) = a$.

$\therefore$ g is onto.

$\Leftarrow$: Assume $\exists g: B \to A$ that is onto.

By 3(b), $g(g^{-1}(A)) = A$, so then $\forall\, b \in g^{-1}(A), g(b) \in A$.

Since g is well-defined, then $b_1 = b_2 \Rightarrow g(b_1) = g(b_2)$.

Define $f: A \to B$ by $f(a) = g^{-1}(a)$ for exactly one $b \in g^{-1}(a)$ using the Axiom of Choice.

Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then $g^{-1}(a_1) = g^{-1}(a_2)$.

Since g is onto and well-defined, then $a_1 = g(g^{-1}(a_1)) = g(g^{-1}(a_2)) = a_2$.

$\therefore$ f is 1-1.

1. (b) Prove that "≤" satisfies the axioms of an order:

(i) card(A) ≤ card(A)

Proof:

Define $f: A \to A$ by $f(x) = x$. Clearly f is 1-1.

Lemma Suppose $B \subset A$ and $\exists\, f: A \to B$ which is 1-1. Then $A \sim B$.

Proof:

Let $C_0 = A \backslash B$. Assume $A \backslash B \neq \varnothing$ as $A \backslash B = \varnothing \Rightarrow A = B$ and there is nothing to prove.

Let $C_1 = f(C_0)$, ..., let $C_{n+1} = f(C_n)$, and let $C = \cup_{n \in \mathbb{N}} C_n$.

Define $h: A \to B$ by $h(z) = \begin{cases} f(z) \text{ if } z \in C \\ z \text{ if } z \notin C \end{cases}$.

To show h is 1-1, let $z_1, z_2 \in A$ such that $h(z_1) = h(z_2)$. We have 3 cases:

(i) $h(z_1) = f(z_1) = f(z_2) = h(z_2)$. Then $z_1 = z_2$ as f is 1-1.

(ii) $h(z_1) = z_1 = f(z_2) = h(z_2)$. This is impossible as

$h(z_1) = z_1 \Rightarrow z_1 \notin C \Rightarrow f(z_2) \notin C \Rightarrow z_2 \notin C \Rightarrow h(z_2) \neq f(z_2)$.

(iii) $h(z_1) = z_1 = z_2 = h(z_2)$. $\therefore$ h is 1-1.

To show h is onto, let $b \in B$. Then $b \in A$ as $B \subset A$.

If $b \in C$, then $\exists\, n \in \mathbb{N}$ such that $b \in C_n$.

Thus, $\exists\, a \in C_{n-1} \subset A$ such that $f(a) = b$, hence $h(a) = b$.

If $b \notin C$, then $h(b) = b$. $\therefore$ h is onto. $\therefore$ $A \sim B$.

(ii) card(A) ≤ card(B) and card(B) ≤ card(A) ⇒ card(A) = card(B).

This is the Schroeder-Bernstein theorem.

Proof:

$\exists\, f: A \to B$ that is 1-1 as card A ≤ card B, and

$\exists\, g: B \to A$ that is 1-1 as card B ≤ card A. So then

$g: B \to g(B)$ is bijective. Thus

$g \circ f: A \to g(B)$ is 1-1. Note that $g(B) \subset A$, so by the lemma above,

$\exists\, k: A \to g(B)$ that is biject. And now we can define

$h: A \to B$ by $h(z) = g^{-1} \circ k(z)$.

To show h is 1-1, let $z_1, z_2 \in A$ such that $h(z_1) = h(z_2)$.

Then $g^{-1} \circ k(z_1) = g^{-1} \circ k(z_2)$. Since $k(z_1), k(z_2) \in g(B)$ and g is bijective, then $k(z_1) = k(z_2)$. Since k is bijective, then $z_1 = z_2$.

To show h is onto, let $b \in B$. Then $g(b) \in g(B)$ and since k is onto, then $\exists\, x \in A$ such that $k(x) = g(b)$.

And, $h(x) = g^{-1}(k(x)) = g^{-1}(g(b)) = b$ as g is 1-1. $\therefore$ card A = card B.

(iii) card(A) ≤ card(B) and card(B) ≤ card(C) ⇒ card(A) ≤ card(C).

Proof:

card(A) ≤ card(B) $\Rightarrow \exists\, f: A \to B$ that is 1-1 and

card(B) ≤ card(C) $\Rightarrow \exists\, g: B \to C$ that is 1-1.

Claim: $g \circ f: A \to C$ is 1-1.

Let $a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$.

Since g is injective, then $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$.

1. (c) Show that
(i) $\mathbb{R} \sim (0, 1)$ Hint: Use $\mathbb{R} \sim (-\pi/2, \pi/2) \sim (0, 1)$ and $f(x) = \arctan x$.

Proof:
Define $g:(0, 1) \to (-\pi/2, \pi/2)$ by $g(x) = \pi(x - 1/2)$.
This function is clearly a bijection, hence the sets are equivalence related.
Define $f: \mathbb{R} \to (-\pi/2, \pi/2)$ by $f(x) = \arctan(x)$.
To show f is injective, let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$. Then $\arctan(x) = \arctan(y)$.
By definition arctan, $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \arctan(y) < \frac{\pi}{2}$. Since arctan is a continuously decreasing function on this interval then $x = y$.
To show f is surjective, let $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
Since $(-\frac{\pi}{2}, \frac{\pi}{2})$ is the range for which arctan is defined, we have $\arctan(\tan y)) = y$.
$\therefore\ \mathbb{R} \sim (-\pi/2, \pi/2) \sim (0, 1)$.

(c) (ii) $(0, 1) \sim [0, 1)$

Proof:

Define $f:(0, 1) \to [0, 1)$ by $f(x) = \begin{cases} \frac{1}{k-1} \text{ if } x = \frac{1}{k} \text{ for } k > 2, k \in N \\ 0 \text{ if } x = \frac{1}{2} \\ x \text{ otherwise} \end{cases}$

Let $A = \{\frac{1}{n} \mid n > 1, n \in \mathbb{N}\}$, $B = \{0\}$, and $C = \{x \in (0, 1) \mid x \neq \frac{1}{n}$ for $n > 1, n \in \mathbb{N}\}\}$.
Note that A, B, and C form a partition of $[0, 1)$ and are pairwise disjoint.
To show f is injective, we let $x, y \in (0, 1)$ such that $f(x) = f(y)$.
Then $f(x)$ and $f(y)$ are both in A, both in B, or both in C.
If $f(x), f(y) \in A$, then $f(x) = \frac{1}{n}$ for some integer $n > 1$. Thus $x = \frac{1}{n+1} = y$.
If $f(x), f(y) \in B$, then $x = 1/2 = y$. If $f(x), f(y) \in C$, then $x = f(x) = f(y) = y$.
To show f is surjective, let $y \in [0, 1)$.
If $y \in A$, then $y = \frac{1}{n}$ for some integer $n > 1$. Thus $\frac{1}{n+1} \in (0, 1)$ and $f(\frac{1}{n+1}) = \frac{1}{n}$.
If $y \in B$, then $1/2 \in (0, 1)$ and $f(1/2) = 0$. If $y \in C$, then $y \in (0, 1)$ and $f(y) = y$.
$\therefore f$ is a bijection.

(iii) $[0, 1] \sim (0, 1)$
Proof:

Define $f:[0, 1] \to (0, 1)$ by $f(x) = \begin{cases} \frac{1}{k+2} \text{ if } x = \frac{1}{k} \text{ for } k \in N \\ \frac{1}{2} \text{ if } x = 0 \\ x \text{ otherwise} \end{cases}$

Verification of bijective property of this function is parallel to part (ii) above.

2. Define card(A) < card(B) $\Leftrightarrow$ card(A) $\leq$ card(B) and card(B) $\nleq$ card(A). In other words, there exists a 1-1 function $f: A \to B$, but none of the functions $f: A \to B$ can be onto.
Define $\mathscr{P}(A)$ = {subsets of A}, including $\varnothing$.
(i) Show that if card(A) = n, then card ($\mathscr{P}(A)$) = 2^n.

Proof:
We will prove the result by induction on n.
If $n = 0$, then $\mathscr{P}(A) = \mathscr{P}(\varnothing) = \{\varnothing\}$ and card $\mathscr{P}(A)$ = card $\{\varnothing\} = 1 = 2^0$.
Let $n \geq 1$ and assume card ($\mathscr{P}(A)$) = 2^n.
Let $A' = A \cup \{x\}$ for some $x \notin A$. By our induction hypothesis, card ($\mathscr{P}(A)$) = 2^n.
So then $\mathscr{P}(A) = \{\varnothing, B_2, \ldots, B_{2^n}\}$.
And $\mathscr{P}(A \cup \{x\}) = \mathscr{P}(A) \cup \{\varnothing \cup \{x\}, B_2 \cup \{x\}, \ldots, B_{2^n} \cup \{x\}\}$.
So then card $\mathscr{P}(A \cup \{x\}) = 2^n + 2^n = 2^{n+1}$. $\therefore$ $\forall\, n \in \mathbb{N}$, card ($\mathscr{P}(A)$) = 2^n.

(ii) Show that card(A) < card($\mathscr{P}(A)$) for any set A.
Proof:
Define $f: A \to \mathscr{P}(A)$ by $f(a) = \{a\}$.
If $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, then $\{a_1\} = \{a_2\}$, hence $a_1 = a_2$.
$\therefore$ f is 1-1. $\therefore$ card(A) $\leq$ card($\mathscr{P}(A)$).
If there is a 1-1 function that is also onto, then card(A) = card($\mathscr{P}(A)$).
Suppose there is an onto and 1-1 function $f: A \to \mathscr{P}(A)$.
Let $C = \{a \in A \mid a \notin f(a)\}$.
Since f is onto and $C \in \mathscr{P}(A)$, then $\exists\, b \in A$ such that $f(b) = C$.
If $b \in C$, then $b \notin f(b) = C$, a contradiction.
If $b \notin C$, then $\sim(b \notin f(b))$, hence $b \in f(b) = C$, another contradiction.
In either case, f fails to be onto.
$\therefore$ There is no onto function from A to $\mathscr{P}(A)$. Thus, card(A) < card($\mathscr{P}(A)$).

2. (iii) Show that card($\mathscr{P}(\mathbb{N})$) = $\mathfrak{c}$ or in another notation $2^{\aleph_0} = \mathfrak{c}$.

Proof:
Let $A \in \mathscr{P}(\mathbb{N})$ and let S be the set of 0, 1 sequences. Note that S is uncountable.
Define $f_A : \mathbb{N} \to \{0, 1\}$ by $f_A(a) = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ if } a \notin A \end{cases}$
Define g: $\mathscr{P}(\mathbb{N}) \to S$ by $g(A) = (f_A(1), f_A(2), f_A(3), \ldots)$.
We will show g is a bijection.
To show g is injective, let $A, B \in \mathscr{P}(\mathbb{N})$ such that $A \neq B$.
Then, without loss of generality, we can assume $\exists\, a \in A$ such that $a \notin B$.
Thus, $1 = f_A(a) \neq f_B(a) = 0$, hence $g(A) \neq g(B)$.
To show g is surjective, let $W \in S$. Then $W = (w_1, w_2, w_3, \ldots)$ where $w_i \in \{0, 1\}$.
Let $B = \{i \in \mathbb{N} \mid w_i = 1\}$. Then $B \in \mathscr{P}(\mathbb{N})$ and $g(B) = W$.
Thus g is a bijection from $\mathscr{P}(\mathbb{N})$ to S, an uncountable set, hence $\mathscr{P}(\mathbb{N})$ is uncountable.