HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

1. Check the properties of the following Cantor type sets:
(a) Divide the interval [0, 1] into 5 equal subintervals and remove the 2nd and 4th of them. Repeat this process with the remaining 3 intervals and so on...

The Cantor Set is closed, compact, perfect and is uncountable. We will check for these properties in the set described above.

Call this set C_1. Then $C_1 = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{(1/2)(5^n-1)} \left(\frac{2k-1}{5^n}, \frac{2k}{5^n} \right)$.

(i) C_1 is closed.
Proof:
If we take $X = [0, 1]$, then $C_1 = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{(1/2)(5^n-1)} \left(\frac{2k-1}{5^n}, \frac{2k}{5^n} \right)$.
Since $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{(1/2)(5^n-1)} \left(\frac{2k-1}{5^n}, \frac{2k}{5^n} \right)$ is a union of open sets, then C_1 is closed.

(ii) C_1 is compact.
Proof:
We first note that C_1 is bounded as $\forall\, x \in C_1$, $C_1 \subset N_1(x)$.
Thus since $C_1 \subset \mathbb{R}$, C_1 is closed, and C_1 is bounded, then C_1 is compact by Theorem 2.41 (*If a set E in $\mathbb{R}^n$ has one of the following 3 properties, is has the other 2: (a) E is closed and bounded; (b) E is compact; Every infinte subset of E has a limit point in E.*)

(iii) C_1 is perfect.
Proof:
We can show that $x \in C_1$ can be represented in base 5 where
$x = 0.a_1a_2...a_ka_{k+1}...$ and each $a_i \in \{0, 2, 4\}$ (similar to proof in lecture notes 10/13/10).
Let $0 < \varepsilon < 1$ be arbitrary, but fixed. Then $\exists\, k \in \mathbb{N}$ such that $1/5^k < \varepsilon$.
Let $x \in C_1$, then $x = 0.a_1a_2...a_ka_{k+1}...$.
Choose $x_\varepsilon = 0.a_1a_2...a_kb_{k+1}a_{k+2}...$ where $b_{k+1} = \begin{cases} 0 \text{ if } a_{k+1} \neq 0 \\ 2 \text{ if } a_{k+1} = 0 \end{cases}$.
Then $0 < d(x, x_\varepsilon) < 1/5^{k+1} < 1/5^k < \varepsilon$.
$\therefore$ $C_1 \cap [(x - \varepsilon, x + \varepsilon)\setminus\{x\}] \neq \varnothing$, hence $C_1' = C_1$. Thus C_1 is perfect.

(iv) C_1 is uncountable.
Proof:
Since elements of C_1 can be represented in base 5 using only the digits 0, 2, and 4, then C_1 is isomorphic to the set of all sequences formed by the digits 0, 2, and 4. Since the set of all sequences formed by using the digits 0 and 2 are a subset of C_1 and we have shown this set is uncountable, then C_1 is uncountable.

1. (b) Again, divide the interval [0, 1] into 5 equal subintervals, but now remove the 2nd, 3rd, and 4th subintervals.
Proof:
Call this set C_2. Then $C_2 = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{5^{n-1}-1} \left(\frac{5k+1}{5^n}, \frac{5k+4}{5^n} \right)$.

The remainder of the proof is parallel to part (a) above, and the results are the same.

HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

2. Read and understand the proof of Theorem 2.28.

(i) Theorem 2.28 Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Proof:

If $y \in E$, then $y \in E \cup E' = \overline{E}$. Suppose $y \notin E$. Let $\varepsilon > 0$. Then $y = \sup E \Rightarrow \exists\, x \in E$ such that $y - \varepsilon < x < y$. So then $N_\varepsilon(y)^* \cap E \neq \varnothing$. $\therefore$ y is a limit point of E.

$\therefore$ In either case, $y \in \overline{E}$.

(ii) Let $S \subset \mathbb{R}$ be bounded above. Suppose that $\sup S = \alpha \notin S$. Show that α is a limit point of S.

Proof:

Let $r > 0$. Then $\alpha \notin S \Rightarrow \exists\, x \in S$ such that $\alpha - r < x < \alpha$. Thus $N_\varepsilon(x)^* \cap S \neq \varnothing$.

3. Read the definition of convex sets from page 31.

Definition We call a set $E \subset \mathbb{R}^n$ convex if $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ whenever $\mathbf{x}, \mathbf{y} \in E$ and $0 < \lambda < 1$.

(a) Show that in $\mathbb{R}^n$, $N_r(\mathbf{x})$ and $\overline{N_r(\mathbf{x})}$ are convex.

Proof:

Let $\mathbf{w}, \mathbf{z} \in N_r(\mathbf{x})$ and let $0 < \lambda < 1$. Then $|\mathbf{w} - \mathbf{x}| < r$ and $|\mathbf{z} - \mathbf{x}| < r$.

Then $d(\lambda\mathbf{w} + (1 - \lambda)\mathbf{z}, \mathbf{x}) = |\lambda\mathbf{w} + (1 - \lambda)\mathbf{z} - \mathbf{x}| = |\lambda(\mathbf{w} - \mathbf{x}) + (1 - \lambda)(\mathbf{z} - \mathbf{x})|$.

By the Theorem 1.37 (e), we have $|\lambda(\mathbf{w} - \mathbf{x}) + (1 - \lambda)(\mathbf{z} - \mathbf{x})| \leq |\lambda(\mathbf{w} - \mathbf{x})| + |(1 - \lambda)(\mathbf{z} - \mathbf{x})|$
$= \lambda|(\mathbf{w} - \mathbf{x})| + (1 - \lambda)|(\mathbf{z} - \mathbf{x})| < \lambda r + (1 - \lambda)r = r$.

$\therefore$ $\lambda\mathbf{w} + (1 - \lambda)\mathbf{z} \in N_r(\mathbf{x})$, hence $N_r(\mathbf{x})$ is convex.

$\overline{N_r(x)}$ is convex can be proved in the same way, just replace $<$ with $\leq$.

3. (b) Show that if $S \subset \mathbb{R}^n$ is convex, then $\overline{S}$ is convex, too.

Proof:

Consider $\mathbf{x}, \mathbf{y} \in \overline{S}$. If $\mathbf{x} \in \overline{S}$, then either $\mathbf{x} \in S$ or $\mathbf{x} \in S'$.

In either case, $\forall\, r > 0$, $N_r(\mathbf{x}) \cap S \neq \varnothing$ and $N_r(\mathbf{y}) \cap S \neq \varnothing$.

Let $r > 0$, let $0 < r' < \min\{r/(2\lambda), r/[2(1 - \lambda)]\}$. Let $\mathbf{p} \in N_{r'}(\mathbf{x})$, and let $\mathbf{q} \in N_{r'}(\mathbf{y})$.

Since S is convex, we have $\lambda\mathbf{p} + (1 - \lambda)\mathbf{q} \in S$.

Note that $|\mathbf{x} - \mathbf{p}| < r' \Rightarrow |\lambda(\mathbf{x} - \mathbf{p})| = \lambda|\mathbf{x} - \mathbf{p}| < \lambda \bullet (r/(2\lambda)) = r/2$ and similarly,

$|\mathbf{y} - \mathbf{q}| < r' \Rightarrow |(1 - \lambda)(\mathbf{y} - \mathbf{q})| = (1 - \lambda)|\mathbf{y} - \mathbf{q}| < (1 - \lambda) \bullet (r/[2(1 - \lambda)]) = r/2$. So then

$$\begin{aligned} |\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} + (\lambda\mathbf{p} - (1 - \lambda)\mathbf{q})| &= |\lambda(\mathbf{x} - \mathbf{p}) + (1 - \lambda)(\mathbf{y} - \mathbf{q})| \\ &\leq |\lambda(\mathbf{x} - \mathbf{p})| + |(1 - \lambda)(\mathbf{y} - \mathbf{q})| \\ &< (1/2)r + (1/2)r = r. \end{aligned}$$

Thus $\forall\, r > 0$, $N_r(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \cap S \neq \varnothing$.

So then $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S$ or $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S'$.

$\therefore$ $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in \overline{S}$.

(c) Is S convex if we suppose that $\overline{S}$ is convex? No.

Proof:

Consider $S = [0, 2] \setminus \{1\}$.

Then $\overline{S} = [0, 2]$ is convex, but $(1/2) \bullet 0 + (1/2) \bullet 2 = 1 \notin S$. Thus S is not convex.

HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

4. Solve exercises 17 and 18 from page 44.
(17) Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in [0, 1]? Is E compact: Is E perfect?
Is E countable? No.
Proof:
Let $x = 0.a_1a_2a_3...a_n...$ be the decimal expansion of $x \in E$ where $a_i \in \{4, 7\}$.
Assume E is countable. Then we can list the members of E as follows:
$x_1 = 0.a_{11}a_{12}a_{13}...a_{1n}...$; $x_2 = 0.a_{21}a_{22}a_{23}...a_{2n}...$; $x_n = 0.a_{n1}a_{n2}a_{n3}...a_{nn}...$
We will construct a decimal expansion, $b = 0.b_1b_2b_3...b_n...$, not in the list:
(1) If $a_{11} = 4$, then $b_1 = 7$, else $b_1 = 4$; (2) If $a_{22} = 4$, then $b_2 = 7$, else $b_2 = 4$;
(n) If $a_{nn} = 4$, then $b_n = 7$, else $b_n = 4$. Thus, b differs from every element in the table in at least one digit. $\therefore$ $b \notin E$. $\therefore$ E is uncountable.

Is E dense in [0, 1]? No.

Proof:
$0.48 \in [0, 1]$, but $N_{0.001}(x) \cap E = \varnothing$ as $0.48 - 0.4\overline{7} = 0.0\overline{2}$ and $0.7\overline{4} - 0.48 = 0.26\overline{4}$.
Thus, 0.48 is not a limit point of E, but $0.48 \in [0, 1]$. $\therefore$ E is not dense.

Is E compact? Yes.

Proof:
We will show E is closed and bounded, hence compact.
E is bounded as $\forall\, x \in E$, $d(x, 1/2) < 1$. To show E is closed, we will assume it is not. Then $\exists\, p$, a limit point of E such that $p \notin E$. Let $p = 0.p_1p_2p_3...p_n...$ be the decimal expansion of p. Since $p \notin E$, then there is a smallest index n such that $p_n \neq 4$ and $p_n \neq 7$.
This gives us 3 cases, (1) $p_n \in \{0, 1, 2, 3\}$, (2) $p_n \in \{5, 6\}$, and (3) $p_n \in \{8, 9\}$.
Case (1). Assume $p_n \in \{0, 1, 2, 3\}$ and let m be the smallest index such that $p_m = 7$, if such an m exists (if no such m exists, then $0 \le 0.p_1p_2p_3...p_n... < 0.\overline{4}$). Thus,
$0.p_1p_2p_3...p_{m-1}4p_{m+1}...p_{n-1}\overline{7} < 0.p_1p_2p_3...p_{m-1}7p_{m+1}...p_{n-1}p_n... = p < 0.p_1p_2p_3...p_{n-1}\overline{4}...$.
Both are neighborhoods of p that contain no element of E.
Case (2) Assume $p_n \in \{5, 6\}$. Then $0.p_1p_2p_3...p_{n-1}4\overline{7} < 0.p_1p_2p_3...p_n... < 0.p_1p_2p_3...p_{n-1}7\overline{4}...$.
Again, we have a neighborhood of p that contains no element of E.
Case (3) Assume $p_n \in \{8, 9\}$ and let t = the smallest index t such that $p_t = 7$, if such a t exists (if no such digit exists, then $0.\overline{7} < 0.p_1p_2p_3...p_n... \le 1$). Thus,
$0.p_1p_2p_3...p_{n-1}\overline{7} < 0.p_1p_2p_3...p_{m-1}4p_{m+1}...p_{n-1}p_n... = p < 0.p_1p_2p_3...p_{m-1}7p_{m+1}...p_{n-1}\overline{4}...$.
Again, a neighborhood of p that contains no element of E.
$\therefore$ p is not a limit point of E, contrary to our assumption.
$\therefore$ If p is a limit point of E, then $p \in E$. $\therefore$ E is closed. $\therefore$ E is compact.

Is E perfect? Yes.

Proof:
Let $p \in E$ where the decimal expansion of p is $0.p_1p_2p_3...p_n...$.
Let $r > 0$. Then $\exists\, n$ such that $3/10^n < r$. Let $p' = 0.p_1p_2p_3...p_nb_{n+1}...$ where $b_{n+1} = 4$ if $p_{n+1} = 7$, and $b_{n+1} = 7$ if $p_{n+1} = 4$. Then $p' \in N_r(p)$, $p' \in E$ and $p' \neq p$.
$\therefore$ p is a limit point of E. $\therefore$ Every point of E is a limit point of E.
$\therefore$ Since E is also closed, a shown above, then E is perfect.

HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

(18) Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

Yes.

Proof:

Consider the set E constructed in the following manner.

First take a closed interval in $\mathbb{R}$ with irrational endpoints call it $A = [\hat{a}, \hat{b}]$.

Since $\mathbb{Q}$ is countable we can let $F = \{q_1, q_2, \ldots, q_n, \ldots\}$ where F is a list of all the rationals between $\hat{a}$ and $\hat{b}$.

Let $A_1 = A \setminus (a_1, b_1)$ where $\hat{a} < a_1 < q_1 < b_1 < \hat{b}$ for $a_1, b_1 \in \mathbb{Q}^c$,

Define A_n as follows:

If q_n was already removed (i.e. contained in an interval previously deleted), let $A_n = A_{n-1}$.

If q_n was not already removed , let $A_n = A_{n-1} \setminus (a_n, b_n)$ where $\hat{a} < a_n < q_n < b_n < \hat{b}$, $a_n, b_n \in \mathbb{Q}^c$, $q_n - a_n < \min_{1 \le i \le n} = \{|q_n - b_i|\}$, and $q_n - b_n < \min_{1 \le i \le n} = \{|q_n - a_i|\}$.

Let $E = \cap_{n \in \mathbb{N}} A_n$. By construction E contains no rational number.

Since each A_n is closed then E is closed.

To show E contains all its limit points, let $p \in E$ and let $\varepsilon > 0$.

Then $\exists\, q_i \in \mathbb{Q}$ such that $p < q_i < p + \varepsilon$. Then $\exists\, (a_i, b_i)$ such that $q_i \in (a_i, b_i)$.

Since $p \in E$ it must be the case that $p < a_i < q_i$.

Since $p < a_i < q_i < p + \varepsilon$, then p is a limit point of E. $\therefore$ E is perfect.

HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

6. Solve exercises 22 and 23 from page 45.
(**22**) A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^n$ is separable. **Hint**: Consider the set of points which have only rational coordinates.
Proof:
Let $n \in \mathbb{N}$. We will show $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$.
Clearly $\mathbb{Q}^n \subset \mathbb{R}^n$.

We know $\mathbb{Q}$ is countable, and by Theorem 2.13 we have that $\mathbb{Q}^n$ is countable as $\mathbb{Q}^n$ is a finite cross product of $\mathbb{Q}$ by itself n times.

To show $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we let $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{p} \notin \mathbb{Q}^n$.
We will show $\mathbf{p}$ is a limit point of $\mathbb{Q}^n$.
Let $\varepsilon > 0$. Then for every $i = 1, 2, ..., n$, $\exists\, q_i \in \mathbb{Q}$ such that $0 < p_i - q_i < \varepsilon/\sqrt{n}$.
Thus $|\mathbf{p} - \mathbf{q}| = \sqrt{\sum_{i=1}^{n}\left(p_i - q_i\right)^2} < \varepsilon$. $\therefore$ $\mathbf{q} = (q_1, q_2, ..., q_n) \in \mathbb{Q}^n$ and $\mathbf{q} \in N_\varepsilon(\mathbf{p})$.
$\therefore$ $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, hence, $\mathbb{R}^n$ is separable.

(**23**) A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable base.
Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.

Proof:
Let (X, d) be a separable metric space.
Then $\exists\, Y$, a countable, dense subset of X.
Let $\{V_\alpha\} = \{N_q(y) \mid y \in Y \text{ and } q \in \mathbb{Q}^+\}$.
We will show $\{V_\alpha\}$ is a countable base for X.
Let $x \in X$. Let G be an open subset of X such that $x \in G$.
Then $\exists\, r > 0$ such that $N_r(x) \subset G$, as G is open.
As Y is dense in X, then every point of X is a limit point of Y or a point of Y or both.
Thus $N_{(1/2)r}(x) \cap Y \neq \varnothing$, hence $\exists\, y \in N_{(1/2)r}(x) \cap Y$.
Recall that for any metric d, d: $X \times X \to \mathbb{R}^+ \cup \{0\}$, so then $\forall\, x, y \in X$, $d(x, y) \in \mathbb{R}^+ \cup \{0\}$.
We know $\mathbb{Q}^+$ is dense in $\mathbb{R}^+ \cup \{0\}$, hence $\exists\, q \in \mathbb{Q}^+$ such that $d(x, y) < q < (1/2)r$.
Let $z \in N_q(y)$. Then $d(x, z) < d(x, y) + d(z, y) < (1/2)r + q < (1/2)r + (1/2)r = r$.
Thus $x \in N_q(y) \subset N_r(x) \subset G$.
Hence $\{V_\alpha\}$ is a base for X.
To show $\{V_\alpha\}$ is countable, note that
φ:$\{V_\alpha\} \to \mathbb{Q} \times Y$ defined by $\varphi(N_q(y)) = (q, y)$ is a bijection.
Since $\mathbb{Q} \times Y$ is a finite product of countable sets, then $\mathbb{Q} \times Y$, hence $\{V_\alpha\}$ is countable.
$\therefore$ Every separable metric space has a countable base.

HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

5. Solve exercises 27 and 28 from page 45.

(**27**) Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.
Suppose $E \subset \mathbb{R}^n$, E is uncountable, and let P be the set of all condensation points of E.

Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable.
Hint: Let $\{V_n\}$ be a countable base of $\mathbb{R}^n$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Proof:
Let $E \subset \mathbb{R}^n$ such that E is uncountable, and let P be the set of all condensation points of E.
To show P is closed, let $x \in P'$ and let $r > 0$. Then $\exists\, y \in N_r(x)^* \cap P$.
Since $y \in P$, then $N_{r-d(x,y)}(y)$ is uncountable, hence$N_{r-d(x,y)}(y) \subset N_r(x) \Rightarrow N_r(x)$ is uncountable. Thus $x \in P$. $\therefore$ P is closed.

To show every point of P is a limit point of P, let $p \in P$ and suppose $p \notin P'$.
Then $\exists\, r > 0$ such that $N_r(p) \cap P = \{p\}$.
We will show that $N_r(p) \cap E$ is the countable union of countable sets, hence $p \notin P$, contrary to our assumption.
Let $y \in N_r(p)^* \cap E$. Thus, $y \notin P$.
Let $\{V_n\}$ be a countable base for $\mathbb{R}^n$ (we know $\{V_n\}$ exists by #22, 23 above).
So then $\exists\, n$ such that $y \in V_n \subset (N_r(p)^* \cap E)$, since $N_r(p)^* \cap E$ is open.
Notice that $V_n \cap P = \varnothing$, hence V_n is at most countable.
And because y was chosen arbitrarily, then $(N_r(p)^* \cap E) \subset \bigcup_{n=1}^{\infty} V_n$, a countable union of countable sets. Thus $N_r(p) \cap E$ is countable, and we have our contradiction.
$\therefore$ Every point of P is a limit point of P, hence P is perfect.

To show that at most countably many points of E are not in P, let
$W = \cup\{V_{n_i} \in \{V_n\} \mid V_{n_i} \cap E \text{ is at most countable}\}$.
We will show $W^c = P$.
Let $x \in P$, then $\forall\, r > 0$, $N_r(x)$ contains uncountably many points of E.
Thus $\forall\, n_i$, $x \notin V_{n_i} \in W$, thus $x \notin W$, hence $x \in W^c$.
Let $y \in W^c$. Let $r > 0$. Then $(N_r(y) \cap E) \not\subset V_{n_i}\ \forall\, n_i$. Thus, $N_r(y) \cap E$ contains uncountably many points. Since r was chosen arbitrarily, then $y \in P$. $\therefore$ $P = W^c$.
Since W is a countable union of countable sets, then W is at most countable.

HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

(28) Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (Corollary: Every countable closed set in $\mathbb{R}^n$ has isolated points.) **Hint**: Use Exercise 27.

Proof:
Let (X, d) be a separable metric space, and let $E \subset X$ be closed.
If E is at most countable, then $E = E \cup \varnothing$ which is the union of a perfect set and a set which is at most countable. Assume E is uncountable.
Let P be the set of all condensation points of E.
Since X is separable, then, by Exercise #27, P is perfect and $P^c \cap E$ is at most countable. Since P is closed, then $P \subset E$. Also $P^c \cap E = E \setminus P$ is at most countable.
$\therefore$ $E = P \cup (E \setminus P)$, which is a union of a perfect set and a set which is at most countable.

7. Let $S \subset \mathbb{R}$. Prove that the set of isolated points of S is at most countable.

Proof:
Let $E \subset S$ such that E is a set of isolated points.
If E is finite, then E is countable. Assume E is infinite.
Since E is a set of isolated points then for each $x \in E$, $\exists\, r_x > 0$ such that $N_{r_x}(x) \cap E = \{x\}$.
Let $t_x = (1/4)\, r_x$. We will show $\{N_{t_x}(x)\}_{x \in E}$ are disjoint.
Let $x \in E$. Let $y \in N_{t_x}(x)$. Let $z \in E$ such that $d(x, z) \geq r_x$.
Then $\exists\, r_z > 0$ such that $N_{r_z}(z) \cap E = \{z\}$. Let $r = \max\{r_x, r_z\}$.
Thus $(1/4)r_z = t_z \leq t_x = (1/4)r_x$.
Then $r \leq d(x, z) \leq d(x, y) + d(y, z) \leq (1/4)r_x + d(y, z) \leq (1/4)r + d(y, z)$.
Thus, $(3/4)r_z \leq (3/4)r \leq d(y, z)$, hence $y \notin N_{t_z}(z)$.
And we have that $\{N_{t_x}(x)\}_{x \in E}$ are disjoint.
Now, by the density property, $\exists\, p_x \in \mathbb{Q}$ such that $x < p_x < x + t_x$. Thus $p_x \in N_{t_x}(x)$.
Define $f : E \to \mathbb{Q}$ by $f(x) = p_x$.
To show f is well-defined and 1-1, we let $x, y \in E$ such that $f(x) = f(y)$.

$$\begin{aligned} \text{Then we have}\quad x = y &\Leftrightarrow f(x) = f(y) \\ &\Leftrightarrow p_x = p_y \\ &\Leftrightarrow N_{t_x}(x) \cap N_{t_y}(y) \neq \varnothing \\ &\Leftrightarrow N_{t_x}(x) = N_{t_y}(y). \end{aligned}$$

Thus, f is 1-1.
So then card $E \leq$ card $\mathbb{Q}$.
$\therefore$ E is countable.

8. Let $S \subset \mathbb{R}$ such that every $x \in S$ has a neighborhood which intersects S in at most countably many points. Show that S is at most countable.

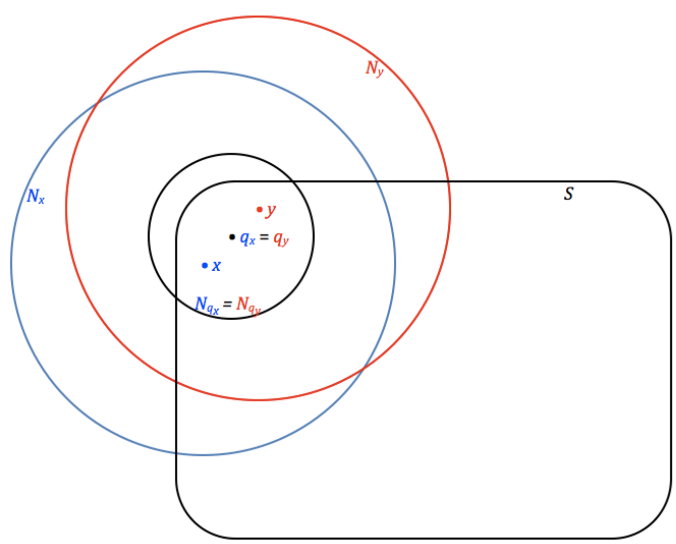
Proof:
We will first construct a set of neighborhoods, D, each neighborhood containing a countable number of points, that covers S. In order to show D, or a subset of D that still covers S (call it D_1 if we need it), is countable we will find a countable set, A, for which we can define a function $f: A \to D$ (or D_1) which is onto. This will give us that D (or D_1) is countable, hence S will be a countable union of countable sets.

Let $x \in S$.
By hypothesis, $\exists\, r > 0$ such that $\mathcal{N}_x = N_r(x) \cap S$ is at most countable.
Let $D = \{\mathcal{N}_x \mid x \in S\}$. This is an open cover of S.
By the density property,
$\exists\, q_x \in \mathbb{Q}$ such that $x < q_x < x + r/4$ and $\exists\, r_{q_x} \in \mathbb{Q}$ such that $d(x, q_x) < r_{q_x} < r/4$.
Let $\mathcal{N}_{q_x} = N_{r_{q_x}}(q_x) \cap S$. Thus $x \in \mathcal{N}_{q_x} \subset \mathcal{N}_x$, and $\mathcal{N}_{q_x}$ is at most countable.
Let $A = \{\mathcal{N}_{q_x} \mid x \in S\}$. Note that A is countable.

Here, we would like to define $f: A \to D$ by $f(\mathcal{N}_{q_x}) = \mathcal{N}_x$,
however, the map is not a well-defined function as it is possible that we could have $\mathcal{N}_{q_x}$ and $\mathcal{N}_{q_y} \in A$ such that $\mathcal{N}_{q_x} = \mathcal{N}_{q_y}$ but $f(\mathcal{N}_{q_x}) \neq f(\mathcal{N}_{q_y})$.
This would occur in the event that $\exists\, y \in S$ such that $y \in \mathcal{N}_{q_x}$ and $\mathcal{N}_x \neq \mathcal{N}_y$ (see diagram).
So then we must reduce D in order to prevent this occurrence.

If $\exists\, y \in S$ such that $y \in \mathcal{N}_{q_x}$ and $\mathcal{N}_x \neq \mathcal{N}_y$, we can reduce D by eliminating $\mathcal{N}_y$ as the remaining neighborhoods in D still cover S. Call this reduced cover D_1.
Now we can define $f: A \to D_1$ by $f(\mathcal{N}_{q_x}) = \mathcal{N}_x$.
This map is well-defined and clearly onto.
$\therefore$ D_1 is a countable open cover of S and each $\mathcal{N}_x \in D_1$ is countable.
Since $S = \cup D_1$, then S is a countable union of countable elements, hence countable.

This diagram is a genralization of the technique used in this problem to $S \subseteq \mathbb{R}^2$.



HW 7, Practice Problems, Week #7, Assigned Friday, 10/15/10

9. Show that (0, 1) × (0, 1) ~ (0, 1).

Proof:
We will show there is an injective function f: (0, 1) × (0, 1) → (0, 1) and an injective function g: (0, 1) → (0, 1) × (0, 1), and thus, by the Schroder-Bernstein Theorem, card ((0, 1) × (0, 1)) = card ((0, 1)).
Let $x = 0.x_1x_2...$, and $y = 0.y_1y_2...$ be unique decimal expansions of $x, y \in (0, 1)$.

Define f: (0, 1) × (0, 1) → (0, 1) by $f((x, y)) = f((0.x_1x_2..., 0.y_1y_2...)) = 0.x_1y_1x_2y_2...$.
To show f is well-defined and 1-1, we let $(a, b), (c, d) \in (0, 1) \times (0, 1)$ such that $f((a, b) = f((c, d))$.
Then we have $f((a, b) = f((c, d))$

$\Leftrightarrow$ $0.a_1b_1a_2b_2... = 0.c_1d_1c_2d_2...$

$\Leftrightarrow$ $a_1 = c_1, b_1 = d_1, a_2 = c_2, b_2 = d_2, ...$

$\Leftrightarrow$ $(a, b) = (c, d)$.

Thus f is well-defined and 1-1,

Define g: (0, 1) → (0, 1) × (0, 1) by $g(x) = (x, 0.5)$. g is clearly 1-1.
∴ card ((0, 1) × (0, 1)) = card ((0, 1)), hence (0, 1) × (0, 1) ~ (0, 1).