HW 8, Practice Problems, Week #8, Assigned Thursday, 10/21/10

1. Comments regarding the proof of Theorem 2.37

Theorem 2.37

If E is an infinite subset of a compact set K, then E has a limit point in K.

(a) Rewrite the proof of Theorem 2.37 using the following hint: Suppose that E does not have limit points. Then E is closed and by Theorem 2.35 E is compact. Build now an open cover of E (not K!) which does not admit a finite subcover.

Proof:

Let E be an infinite subset of a compact set K.

Assume E has no limit point in K.

Then E contains all of its limit points (of which it has none), hence E is closed.

And by Thm 2.35 (*Closed subsets of compact sets are compact.*), E is compact.

Then $\forall\, x \in E$, $\exists\, r_x$ such that $N_{r_x}(x) \cap E = \{x\}$. $\{N_{r_x}(x) \mid x \in E\}$ is an open cover of E.

But E has infinitely many points, hence $\{N_{r_x}(x) \mid x \in E\}$ contains no finite subcover of E, contrary to the compactness of E.

(b) Write a constructive proof of Theorem 2.37 using the following arguments: Cover K with open balls $N_1(x)$ of radius 1. Extract a finite subcover, and choose one of these balls which contains infinitely many elements of E. Name K_1 the intesection of the closure of this ball with K. Repeat the process by covering K_1 with balls of radius 1/2.

Proof:

Let E be an infinite subset of a compact set K.

$\{N_1(x) \mid x \in K\}$ is an open cover of K. Since K is compact,

$\exists\, \{N_1(x_i)\}_{1 \le i \le n} \subset \{N_1(x) \mid x \in K\}$, a finite subcover of K.

Since E has infinitely many points, then $\exists\, i_1$ such that $N_1(x_{i_1})$ has infinitely many points of E.

Let $K_1 = \overline{N_1(x_{i_1})} \cap K$.

By corollary to Thm 2.35 (*If F is closed and K is compact, then F ∩ K is compact.*), K_1 is compact.

$\{N_{1/2}(x) \mid x \in K_1\}$ is an open cover of K_1. Since K_1 is compact,

$\exists\, \{N_{1/2}(x_i)\}_{1 \le i \le n'} \subset \{N_{1/2}(x) \mid x \in K_1\}$, a finite subcover of K_1.

Since K_1 has infinitely many points of E, then $\exists\, i_2$ such that $N_{1/2}(x_{i_2}) \cap K_1$ has infinitely many points of E.

Let $K_2 = \overline{N_{1/2}(x_{i_2})} \cap K_1$.

Continuing the construction in this manner, we have $K_n = \overline{N_{1/n}(x_{i_n})} \cap K_{n-1}$.

Notice that $K_n \subset K_{n-1} \subset \cdots \subset K_1 \subset K$ and $\forall\, n$, K_n contains infinitely many points of E.

Thus, we have $\{K_\alpha\}$, a collection of compact subsets of K such that every finite subcollection of $\{K_\alpha\}$ is nonempty. By Theorem 2.36, $\cap K_\alpha \neq \varnothing$.

Let $w \in \cap K_\alpha$. Then $\forall\, \varepsilon > 0$, $\exists\, n$ such that $1/n < \varepsilon$ and $w \in \overline{N_{1/(n+1)}(x_{i_{n+1}})} \cap K_n$.

Since $\overline{N_{1/(n+1)}(x_{i_{n+1}})} \cap K_n$ contains infinitely many points of E, then

$\exists\, z \in \overline{N_{1/(n+1)}(x_{i_{n+1}})} \cap K_n$ such that $z \in E$ and $z \neq w$. And $d(z, w) < 1/(n+1) < 1/n < \varepsilon$.

Thus $z \in N_\varepsilon(w)$, hence $N_\varepsilon(w)^* \cap E \neq \varnothing$. $\therefore$ w is a limit point of E in K, as desired.

2. (a) Use Ex. 7 from Week #6 to prove the following fact:
Exercise 7: Let (X, d_X) and (Y, d_Y) be metric spaces.
Define $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. Then $(X \times Y, d)$ is a metric space.
Let $U \subset X \times Y$ be an open set.
Define $U_X = \{x \in X: \exists\, y \in Y \text{ such that } (x, y) \in U\}$ as the projection of U onto X.
Then U is open then U_X is open.

If (X, d_X) and (Y, d_Y) are metric spaces and $K \subset X$, $Q \subset Y$ are compact, then $K \times Q$ is compact in $X \times Y$.

Proof:
Let $\{G_\alpha\}$ be an open cover of $K \times Q$.
So then by Exercise 7, Week #6, we have for each α, $G_{\alpha X}$, the projection of G_α onto X is open and for each α, $G_{\alpha Y}$, the projection of G_α onto Y is open. That is,
$G_{\alpha X}, = \{x \in X: \exists\, y \in Y \text{ such that } (x, y) \in G_\alpha\}$ and
$G_{\alpha Y}, = \{y \in Y: \exists\, x \in X \text{ such that } (x, y) \in G_\alpha\}$
We also have that $K \subset \cup\{G_{\alpha\, X}\}$ and $Q \subset \cup\{G_{\alpha\, Y}\}$.
Thus, $\{G_{\alpha X}\}$ and $\{G_{\alpha Y}\}$ are open covers of K and Q respectively.
Since K is compact, then we can extract $\{G_{mX}\}$, a finite open cover of K.
And since Q is compact, we can extract $\{G_{nY}\}$ a finite open cover of Q.
Note that for each i and each j, $G_{i\,X} = G_{\alpha_i X}$ for some α_i and $G_{j\,Y} = G_{\alpha_j Y}$ for some α_j.
So then $K \times Q \subset \bigcup_{i=1}^{n_1} \bigcup_{j=1}^{n_2} G_{\alpha_{ij}}$. Thus $\left\{G_{\alpha_{ij}}\right\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ is a finite subcover of $\{G_\alpha\}$.

$\therefore$ $X \times Y$ is compact.

(b) Use (a) to prove Theorem 2.40
Theorem 2.40 Every k-cell is compact.

Proof:
Let I^n be a k-cell, consisting of all points $\mathbf{x} = (x_1, ..., x_n)$ such that $a_j \le x_j \le b_j$ $(1 \le j \le k)$.
We will show a closed interval, I, is compact, hence by the preceding problem, $I \times I$ is compact. And by induction we have I^n is compact.
Let $I_1 = [a, b]$ for $a < b$, $a, b \in \mathbb{R}$ and suppose I is not compact.
Let $\{G_\alpha\}$ be an open cover of I and let $c_1 = (a + b)/2$.
Then since $[a, c_1] \cup [c_1, b] = [a, b]$, one of the intervals $[a, c_1]$ or $[c_1, b]$ must be non-compact, call it I_2. Subdivide I_2 and continue the process to get closed sets $\{I_n\}$ such that $I_{n+1} \subset I_n \subset \cdots \subset I_2 \subset I_1$ where $\operatorname{diam}(I_n) = (1/2^n) \bullet d(a, b)$ and none of the I_J 's are compact. By Theorem 2.38 $\exists\, w \in \cap I_n$.
Then $w \in G_\alpha$ for some α. Since G_α is open, then $\exists\, r > 0$ such that $N_r(w) \subset G_\alpha$. By the Archimedean Property, $\exists\, n \in \mathbb{N}$ such that $(1/2^n) \bullet d(a, b) < r$, hence $I_n \subset G_\alpha$, a contradiction to our assumption that no I_n is compact.

3. Let (X, d) be a metric space.
For $\varnothing \neq K \subset X$, define the diameter of K as: $\text{diam}(K) = \sup\{d(x, y): x, y \in K\}$.
Consider a sequence $\{K_n\}$ of nonempty compact subsets of X such that :
(i) $K_{n+1} \subset K_n$, $\forall\, n \in \mathbb{N}$.
(ii) $\text{diam}(K_n) \to 0$, as $n \to \infty$.
Show that there exists a unique $x_0 \in X$ such that $\cap_{n\in\mathbb{N}} K_n = \{x_0\}$.

Proof:
By Theorem 2.36 (*If $\{K_\alpha\}$ is a collection of compact sets such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty,then $\cap_\alpha K_\alpha \neq \varnothing$.*), $\cap_{n\in\mathbb{N}} K_n \neq \varnothing$.
Suppose $\exists\, x, y \in \cap_{n\in\mathbb{N}} K_n$.
Let $\varepsilon > 0$, then $\exists\, n \in \mathbb{N}$ such that $\text{diam}(K_n) < \varepsilon$. Thus $d(x, y) < \varepsilon$.
Since $\text{diam}(K_n) \to 0$, as $n \to \infty$, then $\forall\, \varepsilon > 0$, $0 \leq d(x, y) < \varepsilon$.
So then $d(x, y) = 0$, hence, $x = y$.
$\therefore$ There exists a unique $x_0 \in X$ such that $\cap_{n\in\mathbb{N}} K_n = \{x_0\}$.

4. Again, let (X, d) be a metric space, $\varnothing \neq K \subset X$, and $\{G_\alpha\}$ an open cover of K. A number $\varepsilon > 0$ is called a Lebesgue number of the covering $\{G_\alpha\}$ if every $E \subset K$ with $\operatorname{diam}(E) < \varepsilon$, is a subset of some set G_{α_0} of the covering.
Show that if K is compact, then every open cover has a Lebesgue number.

Proof:
Let (X, d) be a metric space, let $\varnothing \neq K \subset X$, and let $\{G_\alpha\}$ be an open cover of K.
Assume $X \notin \{G_\alpha\}$ (for if $X \in \{G_\alpha\}$, then any positive number is a Lebesgue number).
Since K is compact, then we can choose $\{G_1, G_2, ..., G_n\}$ (possibly with reindexing) such that $\{G_1, G_2, ..., G_n\}$ covers K. For each i, let $C_i = X - G_i$ and define $f: X \to \mathbb{R}$ by $f(x) = (d(x, C_1) + d(x, C_2) + \cdots + d(x, C_n))/n$
where $d(x, C) = \inf\{d(x, c) \mid c \in C\}$.
Let $x \in X \cap G_i$ for some i. Since G_i is open, then $\exists\ \varepsilon > 0$ such that $N_\varepsilon(x) \subset G_i$.
Then $d(x, C_i) \geq \varepsilon$. So $f(x) \geq \varepsilon/n$.
Since f is continuous, it has a minimum value δ.
Let $B \subset X$ such that $\operatorname{diam}(B) < \delta$.
Let $x_0 \in B$ and let $d(x_0, C_m) = \sup\{d(x_0, C_i) \mid i = 1, 2, ..., n\}$. Then $B \subset N_\delta(x_0)$.
And $\delta \leq f(x_0) = (d(x_0, C_1) + d(x_0, C_2) + \cdots + d(x_0, C_n))/n \leq (n \bullet d(x_0, C_m))/n = d(x_0, C_m)$.
Then $B \subset N_\delta(x_0) \subset G_m = X - C_m$ of the covering $\{G_\alpha\}$.

5. Let $E \subset \mathbb{R}$ be an uncountable set. Show that E has limit points.

Proof:
As E is uncountable, then $\exists\ a, b \in E$ such that $[a, b]$ contains uncountably many points.
If $[a, (a + b)/2]$ contains uncountably many points let $a_1 = a$ and let $b_1 = (a + b)/2$.
Otherwise, let $(a + b)/2 = a_1$ and let $b = b_1$. Continue subdividing to get $[a_n, b_n]$ contains uncountably many points and $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$.
Then $[a_n, b_n]$ is closed, bounded, nonempty, hence compact, hence $\cap_{n \in \mathbb{N}}[a_n, b_n] \neq \varnothing$.
Let $w \in \cap_{n \in \mathbb{N}}[a_n, b_n]$. So then $\forall\ n, w \in [a_n, b_n]$. Let $r > 0$.
Then $\exists\ n$ such that $(b - a)(1/2^n) < r$. Thus $N_r(w) \cap [a_n, b_n] \neq \varnothing$.
$\therefore$ w is a limit point of E.

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6. Let (X, d) be a metric space.

A nonempty $K \subset X$ is called *totally bounded* if $\forall\ \varepsilon > 0$ there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of K such that $K \subset \cup_{1 \le i \le n} N_\varepsilon(x_i)$.
$K \subset X$ is called *bounded* if K can be included in a ball of finite radius, or if $\mathrm{diam}(K) < +\infty$.

(a) Show that if K is totally bounded, then it is bounded.

Proof:
Assume K is totally bounded.

Then $\forall\ \varepsilon > 0$ there exists a finite subset $\{x_1, x_2, ..., x_n\}$ of K such that $K \subset \cup_{1 \le i \le n} N_\varepsilon(x_i)$.

Let $\varepsilon = 1$, then $K \subset \cup_{1 \le i \le n} N_1(x_i)$ for some $n \in \mathbb{N}$.

Thus $\forall\ x \in K$, $d(x, x_1) \le d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n) < 2n + 2$.

$\therefore$ K is bounded.

(b) Does bounded imply totally bounded? No.

Proof:
Let (X, d) be a metric space where d is the discrete metric.

Let K be an infinte subset of X. Then $\forall\ x \in K$, $N_{1/2}(x) = \{x\}$.

Thus $\cup\{N_{1/2}(x) | x \in K\}$ is an infinite set.

(c) Show that if K is compact, then it is totally bounded.

Proof:
Let K be a compact set. Let $\varepsilon > 0$. Then $\{N_\varepsilon(x) |\ x \in K\}$ is an open cover of K.

Since K is compact, then $\exists\ \{x_1, x_2, ..., x_n\}$ of K such that $K \subset \cup_{1 \le i \le n} N_\varepsilon(x_i)$.

$\therefore$ K is totally bounded.

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7. Let (X, d) be a metric space. $K \subset X$ is called nowhere dense if interior $(\overline{K}) = \varnothing$.
(a) Show that finite sets are nowhere dense in $\mathbb{R}$.
Proof:
Let $E = \{x_1, x_2, ..., x_n\} \subset \mathbb{R}$. By corollary to Theorem 2.20 (A finite set has no limit points), then E contains only isolated points. Thus $\forall\, x_i \in E$.

(b) Give examples of countable subsets of [0, 1] which are nowhere dense and not nowhere dense.
Proof:
Nowhere dense and countable: $\left\{\frac{1}{n} \mid n \in N\right\}, \left\{\frac{1}{2^n} \mid n \in N\right\},$
<u>Not</u> nowhere dense and countable: $[0, 1] \cap \mathbb{Q}$ where $X = \mathbb{Q}$.

(c) Give an example of an uncountable nowhere dense set.
Nowhere dense and uncountable: the Cantor Set.

(d) Consider a sequence $\{E_n\}$ of nowhere dense subsets of [0, 1].
Show that $\cup_{n\in\mathbb{N}} E_n \neq [0, 1]$.
Proof:
For $i = 1, 2, ..., n$, Let $E_i \subset [0, 1]$ such that E_i is nowhere dense.
Then $\exists\, y_1 \in (0, 1)$ and $\exists\, r_1 > 0$ such that $N_{r_1}(y_1) \subset \overline{N_{r_1}(y_1)} \subset (0, 1)$
and $N_{r_1}(y_1) \cap \overline{E}_1 = \varnothing$, as otherwise every point of E^c would be a limit point of E, hence E would be dense in E^c (i.e. $\forall\, y \in E^c$, $\forall\, r > 0$, $N_r(y_1) \cap \overline{E}_1 \neq \varnothing$, hence y is a limit point of $\overline{E}_1$).
And $N_{r_1}(y_1) \not\subset \overline{E}_2 \Rightarrow \exists\, y_2 \in N_{r_1}(y_1)$ and $\exists\, r_2 > 0$ such that $N_{r_2}(y_2) \subset \overline{N_{r_2}(y_2)} \subset \overline{N_{r_1}(y_1)}$
and $N_{r_2}(y_2) \cap \overline{E}_2 = \varnothing$.
We can continue this process and get that
$N_{r_n}(y_n) \not\subset \overline{E}_n \Rightarrow \exists\, y_n \in U_{n-1}$ such that $N_{r_n}(y_n) \subset \overline{N_{r_n}(y_n)} \subset (0, 1)$
and $N_{r_n}(y_n) \cap \overline{E}_n = \varnothing$.
Since each of the $\overline{U}_i$ is closed and bounded, then each is compact.
So then by Theorem 2.37 (*If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supseteq K_{n+1}$ for $n = 1, 2, ...,$ then* $\bigcap_1^\infty K_n$.) we have that $\cap \{\overline{U}_n \mid n \in \mathbb{N}\} \neq \varnothing$.
Thus $\exists\, x \in \cap \{\overline{U}_n \mid n \in \mathbb{N}\}$ such that $x \notin \cup_{n\in\mathbb{N}} E_n$.
$\therefore\ \cup_{n\in\mathbb{N}} E_n \neq [0, 1]$.