HW 9, Practice Problems, Week #9, Assigned Friday, 10/29/10

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

Proof:
Let A and B be disjoint closed sets. Then $\overline{A} \cap \overline{B} = \varnothing$. Since $A \subset \overline{A}$ and $B \subset \overline{B}$, then $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$. $\therefore$ A and B are separated.

(b) Prove the same for disjoint open sets.

Proof:
Let A and B be disjoint open sets. Then $A \cap B = \varnothing$.
Suppose $\overline{A} \cap B \neq \varnothing$. Then $\exists\, x \in \overline{A} \cap B$. Since $A \cap B = \varnothing$, then $x \in A'$ and $x \in B$.
Since B is open, then $\exists\, r > 0$ such that $N_r(x) \subset B$. But $A \cap B = \varnothing \Rightarrow A \cap N_r(x) = \varnothing$.
So then we have a neighborhood of x whose intersection with A is empty. This contradicts that $x \in A'$. $\therefore$ $\overline{A} \cap B = \varnothing$. The proof for $A \cap \overline{B} = \varnothing$ is similar.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with > is place of <. Prove that A and B are separated.

Proof:
Let $p \in X$, let $\delta > 0$, let $A = \{q \in X \mid d(p, q) < \delta\}$, and let $B = \{q \in X \mid d(p, q) > \delta\}$.
Note that $\overline{A} = \{q \in X \mid d(p, q) \leq \delta\}$ and $\overline{B} = \{q \in X \mid d(p, q) \geq \delta\}$.
If $x \in A$, then $d(p, x) \ngeq \delta$, hence $x \notin \overline{B}$.
And if $x \in B$, then $d(p, x) \ngeq \delta$, hence $x \notin \overline{A}$.
$\therefore$ $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$, hence A and B are separated.

(d) Prove that every connected metric space with at leat two points is uncountable.
Hint: Use (c).

Proof:
Let X be a connected metric space. Let $a, b \in X$ such that $a \neq b$.
Then $r_0 = d(a, b) > 0$.
For each $r < r_0$, let $A_r = \{x \in X \mid d(x, a) > r\}$ and $B_r = \{x \in X \mid d(x, a) < r\}$.
Since $r < r_0$, then $b \in A_r$, hence $A_r \neq \varnothing$; and since $a \in B_r$, then $B_r \neq \varnothing$.
Note that $A_r \cap B_r = \varnothing$.
Let $C_r = \{x \in X \mid d(x, r) = r\}$. We know $C_r \neq \varnothing$ for if it did,
then we would have $X = A_r \cup B_r$, a union of nonempty, separated sets.
Let $Y = \cup\{C_r\}$, and let $Z = \{r \in \mathbb{R} \mid 0 < r < r_0\}$.
Define $\varphi : Y \to Z$ by $\varphi(y) = d(y, a)$.
This is a function from $Y \subseteq X$ onto $(0, r_0)$, an uncountable set.
Thus, Y, hence X is uncountable.

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20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$)

Closures of connected sets are always connected, but not interiors.

Proof:

Let S be a connected set of a metric space X.

Suppose $\overline{S}$ is not connected. Then $\overline{S} = A \cup B$ where $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$.

Since S is connected, we must have $S \neq A \cup B$ as otherwise S would be separated by A and B as well. So then $\overline{S} = A \cup B \Rightarrow S \subset A \cup B$, hence $S = (A \cap S) \cup (B \cap S)$.

Note that $\overline{B \cap S} \subset \overline{\overline{B} \cap \overline{S}} = \overline{B} \cap \overline{S}$.

Thus, $(A \cap S) \cap (\overline{B \cap S}) \subset (A \cap S) \cap (\overline{B} \cap \overline{S}) = A \cap \overline{B} \cap S = \varnothing \cap S = \varnothing$.

Similarly $\overline{A \cap S} \cap (B \cap S) \subset \varnothing$.

If both $(A \cap S)$ and $(B \cap S)$ are nonempty, then these sets form a separation of S.

Thus, $A \cap S = \varnothing$ or $B \cap S = \varnothing$. Without loss of generality, assume $A \cap S = \varnothing$.

Then $S \subset B$ which implies that $\overline{S} \subset \overline{B}$.

Since $A \cap \overline{B} = \varnothing$, then $A = A \cap \overline{S} \subset A \cap \overline{B} = \varnothing$.

Since $A = \varnothing$, then $\overline{S}$ must be connected.

Counterexample to show interiors of connected sets are not always connected:

Let A = $\{(x, y) \in \mathbb{R}^2 | \ (x+1)^2 + y^2 \leq 1\}$ and let B = $\{(x, y) \in \mathbb{R}^2 | \ (x-1)^2 + y^2 \leq 1\}$. These sets are connected at the origin. However

A° = $\{(x, y) \in \mathbb{R}^2 | \ (x+1)^2 + y^2 < 1\}$ and B° = $\{(x, y) \in \mathbb{R}^2 | \ (x-1)^2 + y^2 < 1\}$ are disconnected.

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21. Let A and B be separated subsets of some $\mathbb{R}^n$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define $\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$ for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [i.e. $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]
(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.

Proof:
Let A and B be separated subsets of some $\mathbb{R}^n$, then $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.
Let $\mathbf{a} \in A$, and let $\mathbf{b} \in B$. Since A and B are separated, then $\mathbf{a} \neq \mathbf{b}$.
If $A_0 \cap \overline{B}_0 \neq \varnothing$, then $\exists\, x \in A_0$ such that $\forall\, r > 0$, $N_r(x) \cap B_0 \neq \varnothing$.
And $x \in A_0 \Rightarrow \mathbf{p}(x) \in A$. Since $A \cap \overline{B} = \varnothing$, then $\mathbf{p}(x) \notin \overline{B}$.
So then $\exists\, r > 0$ such that $N_r(\mathbf{p}(x))^* \cap B = \varnothing$.
Let $r' = r/|\mathbf{b} - \mathbf{a}|$. Then $\exists\, t \in N_{r/|\mathbf{b}-\mathbf{a}|}(x)$ such that $t \neq x$. Thus
$0 < |x - t| < \dfrac{r}{|\mathbf{b} - \mathbf{a}|} \Rightarrow |x - t| \bullet |\mathbf{b} - \mathbf{a}| < r \Rightarrow |(\mathbf{a} - x\mathbf{a} + x\mathbf{b}) - (\mathbf{a} - t\mathbf{a} + t\mathbf{b})| < r$
$\Rightarrow |((1 - x)\mathbf{a} + x\mathbf{b}) - ((1 - t)\mathbf{a} + t\mathbf{b})| < r \Rightarrow |\mathbf{p}(x) - \mathbf{p}(t)| < r \Rightarrow \mathbf{p}(t) \in N_r(\mathbf{p}(x)) \cap B$,
contrary to $N_r(\mathbf{p}(x))^* \cap B = \varnothing$.
Thus $N_{r/|\mathbf{b}-\mathbf{a}|}(x)^* \cap B_0 = \varnothing$, hence $x \notin \overline{B}_0$. $\therefore$ $A_0 \cap \overline{B}_0 = \varnothing$.
By very similar proof we have that $\overline{A}_0 \cap B_0 = \varnothing$.
$\therefore$ A_0 and B_0 are separated.

(b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin (A \cup B)$.

Proof:
Assume there is no such t_0. Let $I^\circ = (0, 1)$.
Then $\forall\, t_0 \in I^\circ$, $\mathbf{p}(t_0) \in (A \cup B)$, that is $\mathbf{p}(I^\circ) \subset (A \cup B)$,
Thus, $I^\circ \subset \mathbf{p}^{-1}(\mathbf{p}(I^\circ)) \subset \mathbf{p}^{-1}(A \cup B) = \mathbf{p}^{-1}(A) \cup \mathbf{p}^{-1}(B) = A_0 \cup B_0$.
Then we have $I^\circ = [A_0 \cap I^\circ] \cup [B_0 \cap I^\circ]$.
And since by part (a) $A_0 \cup B_0$ are separated, then we have
$[A_0 \cap I^\circ] \cup \overline{[B_0 \cap I^\circ]} \subset [A_0 \cap I^\circ] \cup [\overline{B}_0 \cap [0,1]] = A_0 \cap \overline{B}_0 \cap [0,1] = \varnothing$, and
$\overline{[A_0 \cap I^\circ]} \cup [B_0 \cap I^\circ] \subset [\overline{A}_0 \cap I^\circ] \cup [B_0 \cap [0,1]] = \overline{A}_0 \cap B_0 \cap [0,1] = \varnothing$.
Thus $I^\circ \subset A_0$ or $I^\circ \subset B_0$ as otherwise $I^\circ = [A_0 \cap I^\circ] \cup [B_0 \cap I^\circ]$, a union of nonempty separated sets (yet we know (0, 1) is connected).
If $I^\circ \subset A_0$, then $[0, 1] \subset \overline{A}_0$.
But $\mathbf{p}(1) = (1-1)\mathbf{a} + 1\mathbf{b} = \mathbf{b} \in B$, hence $\mathbf{p}^{-1}(1) \in B_0$, contrary to $\overline{A}_0 \cap B_0 = \varnothing$.
If $(0, 1) \subset B_0$, then $[0, 1] \subset \overline{B}_0$.
But $\mathbf{p}(0) = (1-0)\mathbf{a} + 0\mathbf{b} = \mathbf{a} \in A$, hcontrary to $A_0 \cap \overline{B}_0 = \varnothing$.
$\therefore$ There must exist $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin (A \cup B)$.

(c) Prove that every convex subset of $\mathbb{R}^n$ is connected.

Proof:
Let E be a convex subset of $\mathbb{R}^n$. Then $\forall\, \mathbf{x}, \mathbf{y} \in E$, $\forall\, t \in (0, 1)$, $(1 - t)\mathbf{x} + t\mathbf{y} \in E$.
Suppose $\exists\, A \subset E$, $B \subset E$ such that $E = A \cup B$, $A \cap \overline{B} = \varnothing$, and $\overline{A} \cap B = \varnothing$.
Then by part (b), $\exists\, t_0 \in (0, 1)$ such that $(1 - t)\mathbf{x} + t\mathbf{y} \notin E$ for some $\mathbf{x} \in A$ and $\mathbf{y} \in B$, contrary to the definition of convex.

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30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Equivalent Statement:

If G_i is a dense open subset of $\mathbb{R}^n$, for i = 1, 2, 3, ..., then $\bigcap_{i=1}^{\infty} G_i \neq \varnothing$ (in fact it is dense in $\mathbb{R}^n$).

(*This is a special case of Baire's theorem; see Exercise 22, chapter 3 for the general case.*)

Proof:

Let $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed subset of $\mathbb{R}^n$,

To show at least one F_n has a nonempty interior we will suppose by contradiction that $\forall\ n \in \mathbb{N}$, F_n all have empty interiors. Since F_1 has empty interior, then $\mathbb{R} \not\subset F_1$, hence $F_1{}^c$ is an open, nonempty subset of $\mathbb{R}^k$.

Thus $\exists\ x_1 \in F_1{}^c$ and $\exists\ r_1$ such that $N_{r_1}(x_1) \subset F_1{}^c$, as otherwise $\forall$ x $\in F_1{}^c$ and $\forall$ r > 0, $N_{r_1}(x_1) \cap F_1 \neq \varnothing$, hence every point of $F_1{}^c$ would be a limit point of F_1.

Let $V_1 = N_{1/2(r_1)}(x_1)$. Then $\overline{V_1} = \overline{N_{(1/2)r_1}(x_1)} \subset F_1{}^c$.

For $n > 1$, choose V_n so that $V_n \cap F_n = \varnothing$ (i.e. $V_n \subset F_n{}^c$) and $\overline{V_n} = \overline{N_{(1/2)r_n}(x_n)} \subset V_{n-1}$.

Thus, $\forall\ n \in \mathbb{N}$, $\overline{V}_{n+1} \subset \overline{V}_n$, $\overline{V}_n \neq \varnothing$, and $\overline{V}_n$ is closed, bounded, and hence compact.

So by Corollary to Theorem 2.36 (*If {Kα} is a sequence of nonempty compact sets such that $K_{n+1} \subset K_n$, then $\cap_{n\in\mathbb{N}} K_n \neq \varnothing$.*}, we have that $\cap_{n\in\mathbb{N}} \overline{V}_n \neq \varnothing$.

And $w \in \cap_{n\in\mathbb{N}} \overline{V}_n \Rightarrow \forall\ n \in \mathbb{N}$, $w \in \overline{V}_n \Rightarrow w \notin F_n \Rightarrow w \notin \mathbb{R}^k$, contrary to $\cap_{n\in\mathbb{N}} \overline{V}_n \subset \mathbb{R}^k$.

$\therefore$ At least one F_n has a nonempty interior.