1. Show that in a metric space (X, d), $\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} d(x_n, x) = 0$.

Proof:
We have $\lim_{n\to\infty} x_n = x$
$\Leftrightarrow \forall\ \varepsilon > 0, \exists\ N_\varepsilon \in \mathbb{N}$ such that $\forall\ n > N_\varepsilon$, $d(x_n, x) < \varepsilon$
$\Leftrightarrow \forall\ \varepsilon > 0, \exists\ N_\varepsilon \in \mathbb{N}$ such that $\forall\ n > N_\varepsilon$, $|d(x_n, x) - 0| < \varepsilon$
$\Leftrightarrow \lim_{n\to\infty} d(x_n, x) = 0$.

QED

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2. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences of real numbers. Suppose that $x_n \le y_n \le z_n$, $\forall\ n \in \mathbb{N}$ and that $x_n \to \alpha$ and $z_n \to \alpha$. Show that $y_n \to \alpha$ as $n \to \infty$.

Proof:
Let $\varepsilon > 0$. Then
$x_n \to \alpha \Rightarrow \exists\ N_\varepsilon \in \mathbb{N}$ such that $\forall\ n > N_\varepsilon$, $|x_n - \alpha| < \varepsilon$ and
$z_n \to \alpha \Rightarrow \exists\ N_\varepsilon' \in \mathbb{N}$ such that $\forall\ n > N_\varepsilon'$, $|z_n - \alpha| < \varepsilon$.
Let $N = \max\{N_\varepsilon, N_\varepsilon'\}$. Then $\forall\ n > N$, we have $-\varepsilon < x_n - \alpha < \varepsilon$ and $-\varepsilon < z_n - \alpha < \varepsilon$.
This gives us that $\alpha - \varepsilon < x_n$ and $z_n < \alpha + \varepsilon$. Since $x_n \le y_n \le z_n$, $\forall\ n \in \mathbb{N}$, then
$\alpha - \varepsilon < x_n \le y_n \le z_n < \alpha + \varepsilon$. Thus $-\varepsilon < y_n - \alpha < \varepsilon$, hence $|y_n - \alpha| < \varepsilon$.
$\therefore\ y_n \to \alpha$ as $n \to \infty$.

QED

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3. Let $\{x_n\} \subset X$ and $\{\varepsilon_n\} \subset \mathbb{R}$, $\varepsilon_n \ge 0$, $\forall\ n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$ and $d(x_n, x) \le \varepsilon_n$, $\forall\ n \in \mathbb{N}$. Show that $\lim_{n\to\infty} x_n = x$.

Proof:
Let $\varepsilon > 0$. Then $\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \exists\ N_\varepsilon \in \mathbb{N}$ such that $\forall\ n > N_\varepsilon$, $0 < \varepsilon_n < \varepsilon$.
Thus $d(x_n, x) \le \varepsilon_n < \varepsilon$. $\therefore\ \lim_{n\to\infty} x_n = x$.

QED

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4. Let $\{x_n\}, \{y_n\} \subset X$. Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$. Show that $\lim_{n\to\infty} y_n = x$.

Proof:
$\lim_{n\to\infty} x_n = x \Rightarrow \exists\ N_\varepsilon \in \mathbb{N}$ such that $\forall\ n > N_\varepsilon$, $d(x_n, x) < \varepsilon/2$ and
$\lim_{n\to\infty} d(x_n, y_n) = 0 \Rightarrow \exists\ N_\varepsilon' \in \mathbb{N}$ such that $\forall\ n > N_\varepsilon'$, $|d(x_n, y_n) - 0| = d(x_n, y_n) < \varepsilon/2$.
Let $N = \max\{N_\varepsilon, N_\varepsilon'\}$.
Then $\forall\ n \ge N$, we have $d(y_n, x) \le d(y_n, x_n) + d(x_n, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ by the triangle inequality.
$\therefore\ \lim_{n\to\infty} y_n = x$.

QED

5. Show that if $\{x_n\} \subset X$ has two subsequences that converge to different limits, then $\{x_n\}$ is divergent.

Proof:

Suppose $\{x_{p_k}\}$ and $\{x_{q_j}\}$ are subsequences of $\{x_n\}$ that converge to p and q, respectively, $p \neq q$. And suppose $x_n \to x$. Let $\varepsilon = d(p, q)/4$. Then

$\exists K_\varepsilon \in \mathbb{N}$ such that $\forall k > K_\varepsilon$, $d(x_{p_k}, p) < \varepsilon$ and

$\exists J_\varepsilon \in \mathbb{N}$ such that $\forall j > J_\varepsilon$, $d(x_{q_j}, p) < \varepsilon$.

Then $\exists\ \varepsilon = d(p, q)/4$ such that $\forall N > \max\{ n_{K\varepsilon}, n_{J\varepsilon}\}$ $\exists\ p_k > N$ and $\exists\ q_j > N$ such that

$4\varepsilon = d(p, q) \leq d(x_{p_k}, p) + d(x_{p_k}, x_{q_j}) + d(x_{q_j}, q) < \varepsilon + d(x_{p_k}, x_{q_j}) + \varepsilon$.

So then we have $2\varepsilon < d(x_{p_k}, x_{q_j}) \leq d(x_{p_k}, x) + d(x_{q_j}, x)$.

Thus either $d(x_{p_k}, x) > \varepsilon$ or $d(x_{q_j}, x) > \varepsilon$. Hence $\{x_n\}$ is divergent.

QED

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6. Let $\{x_n\} \subset \mathbb{R}$ such that $x_n > 0$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$. Show that if $L < 1$ then $\lim_{n\to\infty} x_n = 0$. Show that if $L > 1$ then $x_n \to +\infty$. What about the case $L = 1$?

Proof:

Case 1: $L < 1$. Let $L < a < 1$. Then

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L \Rightarrow \exists\ \mathrm{N} \in \mathbb{N}$ such that $\forall n > N$ we have $\left|\frac{x_{n+1}}{x_n} - L\right| < a - L$.

So then $L - a < \frac{x_{n+1}}{x_n} - L < a - L$. Or equivalently, $2L - a < \frac{x_{n+1}}{x_n} < a$.

Thus $\forall n > N$, $x_{n+1} < ax_n$.

So now we have $x_{n+1} < ax_n < a^2x_{n-1} < a^3x_{n-2} < \cdots < a^{n-N+1}x_{n-(n-N)}$.

And this gives us that $x_n < a^{n-N}x_N$.

Since $0 < a < 1$ and x_N is a constant, then $a^{n-N}x_n \to 0$ as $n \to \infty$.

$\therefore$ $x_n \to 0$ as $n \to \infty$.

Case 2: $L > 1$. Let $\{y_n\} = \frac{1}{x_n}$. Then $\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{x_{n+1}}}{\frac{1}{x_n}} = \frac{1}{\lim_{n\to\infty} \frac{x_{n+1}}{x_n}} = \frac{1}{L} < 1$.

By case 1, we have that $\lim_{n\to\infty} y_n = 0$. $\therefore$ $x_n \to +\infty$.

Case 3: $L = 1$. Inconclusive. If $x_n \to p$, then $x_{n+1} \to p$. Thus $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{p}{p} = 1$.

QED

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7. Let $0 \le a \le b$. Show that $\sqrt[n]{a^n + b^n} \to b$.

Proof:

$0 < a < b \Rightarrow 0 < b = \sqrt[n]{b^n} < \sqrt[n]{a^n + b^n} < \sqrt[n]{b^n + b^n} = \sqrt[n]{2}\,\sqrt[n]{b^n} = \sqrt[n]{2}\,b$.

By Theorem 3.20 (b) (*If* $p > 0$, *then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$) we have $\lim_{n\to\infty} \sqrt[n]{2}\,b = b$.

Thus, $\lim_{n\to\infty} b \le \lim_{n\to\infty} \sqrt[n]{a^n + b^n} \le \lim_{n\to\infty} \sqrt[n]{2}b = b$. Thus, $\sqrt[n]{a^n + b^n} \to b$.

QED

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8. Let $\{x_n\} \subset \mathbb{R}^n$. Show that if $x_n \to x$ then $|x_n| \to |x|$. Does $|x_n| \to |x|$ imply $x_n \to x$? Show that if $|x_n| \to 0$ then $x_n \to 0$.

Proof:

Assume $x_n \to x$. Then $\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N}$ such that $\forall\ n \ge N$, $|x_n - x| < \varepsilon$.

So then $||x_n| - |x|| \le |x_n - x| < \varepsilon$. Hence $\{|x_n|\}$ converges.

The converse is not true.

Proof:

Let $x_n = (-1)^n$. $\{|x_n|\}$ converges to 1, but $\{x_n\}$ does not converge.

Proof:

Assume $|x_n| \to 0$. Then $\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N}$ such that $\forall\ n \ge N$, $|x_n - 0| = ||x_n| - 0| < \varepsilon$.

$\therefore\ x_n \to 0$.

QED

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9. Let $\{x_n\}$ be such that $x_n \to +\infty$. Show that $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

Proof:

$x_n \to +\infty \Rightarrow\ \forall\ M > 0\ \exists\ N_M \in \mathbb{N}$ such that $\forall\ n \ge N_M$ we have $x_n > M$. Thus $\frac{1}{x_n} < \frac{1}{M}$.

Note that $x_n \to +\infty$ also implies $\exists\ N_0 \in \mathbb{N}$ such that $\forall\ n \ge N_0$, $x_n > 0$, hence $1/x_n > 0$.

Let $\varepsilon = 1/M$.

Let $N_\varepsilon = \max\{N_M, N_0\}$. Then $\forall\ n \ge N_\varepsilon$, $0 < \frac{1}{x_n} < \frac{1}{M}$. Thus, $\left|\frac{1}{x_n}\right| < \frac{1}{M} = \varepsilon$.

$\therefore\ \lim_{n\to\infty} \frac{1}{x_n} = 0$.

QED

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10. Let $\{x_n\} \subset \mathbb{R}$ such that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to the same number. Show that $\{x_n\}$ is convergent.

Proof:

Suppose $x_{2n} \to x$ and $x_{2n+1} \to x$. Let $\varepsilon > 0$. Then

$\exists\, N_1 \in \mathbb{N}$ such that $\forall\, n \geq N_1$, $|x_{2n} - x| < \varepsilon$ and

$\exists\, N_2 \in \mathbb{N}$ such that $\forall\, n \geq N_2$, $|x_{2n+1} - x| < \varepsilon$.

Let $N_\varepsilon = \max\{N_1, N_2\}$. Since $\{2n \mid n \in \mathbb{N}\} \cup \{2n + 1 \mid n \in \mathbb{N}\} = \mathbb{N}$, then

$\forall\, n \geq N_\varepsilon$, $|x_n - x| < \varepsilon$. $\therefore$ $\{x_n\}$ is convergent.

QED ❧

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11. Let $\{x_n\}$ and $\{y_n\}$ be real sequences. Suppose that $\{x_n\}$ converges to a non-zero number and that $\{x_n \bullet y_n\}$ is convergent. Show that $\{y_n\}$ is convergent too.
What if $x_n \to 0$?

Proof:

Suppose $x_n \to x \neq 0$ and $x_n \bullet y_n \to z$.

Since $x_n \bullet y_n \to z$, then

by Theorem 3.2 (c) (*If $\{p_n\}$ is a convergent sequence in a metric space X, then $\{p_n\}$ is bounded.*)

$|x_n y_n|$ is bounded, by say, M.

Since $x_n \to x$, then $|x_n|$ is bounded, by say, T.

Thus $|x_n y_n| = |x_n||y_n| \leq T|y_n| \leq M$. Thus $|y_n| \leq M/T$.

Let $\varepsilon > 0$. Then $\exists\, N_x \in \mathbb{N}$ such that $\forall\, n \geq N_x$, $|x_n - x| < T\varepsilon|x|/(2M)$.

And $\exists\, N_{xy} \in \mathbb{N}$ such that $\forall\, n \geq N_{xy}$, $|x_n y_n - z| < \varepsilon|x|/(2)$.

Let $N' = \max\{N_x, N_{xy}\}$. Then $\forall\, n \geq N'$,

$|x||y_n - z/x| \leq |y_n||x_n - x| + |x_n y_n - z| < (M/T)T\varepsilon|x|/(2M) + \varepsilon|x|/2 = |x|\varepsilon$.

$\therefore |y_n - z/x| < \varepsilon$, hence $\{y_n\}$ is convergent too.

If $x_n \to 0$, then we don't know if $\{y_n\}$ is convergent.

Proof:

Consider $x_n = \dfrac{1}{n}$ and $y_n = n$. Then $x_n \bullet y_n \to 1$ while $\{y_n\}$ diverges.

And consider $x_n = \dfrac{1}{n}$ and $y_n = 1$. Then $x_n \bullet y_n \to 0$ while $y_n \to 1$.

QED ❧
