Content:

Corollary	$\forall\ a, b > 0, \forall\ n \in$ N, $(ab)^{1/n} = a^{1/n} \bullet b^{1/n}$.
Decimals	$x = \sup E$ where $E = \{n_0 + n_1/10 + n_2/100 + \cdots + n_k/10^k\}$ $k \in \{0, 1, ...\}$
Definition	Extended Real Number System
Defintion	Complex number operations
Aside	Matrix representation of complex numbers
Definition	$\mathbb{R}^n$.

Review

Theorem 1.21

$\forall\ x > 0, \forall\ n \in \mathbb{N}, \exists\ y > 0$ such that $y^n = x$. Notation: $y = x^{1/n} = \sqrt[n]{x}$.

$y = \sup E$ where $E = \{t > 0 |\ t^n < x\}$

We must have $y^n < x$.

If $y^n < x$, then we can construct $h > 0$ such that $y + h \in E$ contrary to $y = \sup E$.

If $y^n < x$, then $\exists\ h > 0$ such that $y - h \notin E$, contrary to $y = \sup E$.

Corollary

$\forall\ a, b > 0, \forall\ n \in \mathbb{N}, (ab)^{1/n} = a^{1/n} \bullet b^{1/n}$.

Proof:

Let $\alpha = a^{1/n}$, and let $\beta = b^{1/n}$. So $\alpha^n = a$, and $\beta^n = b$.

So $ab = \alpha^n\beta^n = (\alpha\beta)^n$ by commutativity of $\bullet$.

So $(ab)^{1/n} = [(\alpha\beta)^n]^{1/n} = a^{1/n} \bullet b^{1/n}$.

Decimals

1.22

Let $x > 0$. Let $A = \{n \in \mathbb{N} \cup \{0\} |\ n \leq x\}$.

$0 \in A$, so $A \neq \varnothing$.

By the Archimedean Property, $\exists\ m \in \mathbb{N}$ such that $m \bullet 1 > x$, so A is bounded above by m.

Thus, $n_0 = \sup A$ exists.

Since A is finite, then $n_0 \in A$.

Hence n_0 is the largest integer such that $n_0 \leq x$.

So then $n_0 \leq x < n_0 + 1$, which gives us $0 \leq x - n_0 < 1$.

Similar to finding n_0, we can find the largest integer n_1 such that $n_1 \leq (x - n_0) \bullet 10 < 10$, and so $n_1/10 < x - n_0 < 1$.

Continuing the process, $\forall k$, we can find $n_0, n_1, n_2, ..., n_k$ such that $n_k \leq (x - n_0 - n_1/10 - n_2/10^2 - \cdots - n_{k-1}/10^{k-1})10^k < 10^k$. So then $x \geq n_0 + n_1/10 + n_2/10^2 + \cdots + n_k/10^k$. Thus $x = \sup \{n_0 + n_1/10 + n_2/10^2 + \cdots + n_k/10^k\}$ $k \in \{0, 1, ...\}$

So $\forall\ p \in \mathbb{Q}$, we have $p = n_0 . n_1n_2n_2...$, a terminating decimal or non-terminating, repeating.

The Extended Real Number System

Definition $\mathbb{R} \cup \{-\infty, \infty\} = \overline{\mathbb{R}}$

The order is fine, but $\overline{\mathbb{R}}$ is not a field.

Examples $\lim_{n\to\infty} 1 + t - 100 = \infty$, demonstrates that $\infty + x = \infty \not\Rightarrow x = 0$.
$\lim_{n\to\infty} x^3 - x^2 = \lim_{n\to\infty} x^3(1 - 1/x) = \infty$, demonstrates $\infty - \infty \neq 0$.
$\lim_{n\to\infty} (\sqrt{x+2} - \sqrt{x}) = \lim_{n\to\infty} (\frac{2}{\sqrt{x+2}+\sqrt{x}}) = 0$, so $\infty - \infty \neq 0$.

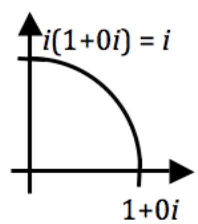
Complex Numbers

Unlike $\overline{\mathbb{R}}$, $\mathbb{C}$ is a field , but it has problems with order.
$\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$ and $\mathbb{C} = \{(x, y) | x, y \in \mathbb{R}\}$
But, the operations in C are defined differently than those in R.

Definition $\forall$ (a, b), (c, d) $\in$ C,
"+": (a, b) + (c, d) = (a + c, b + d)
"•": (a, b) • (c, d) = (ac – bd, ad + bc)

Notation (a, b) = a + ib where i = (0, 1),
Note that $i^2 = -1$ as (0, 1) • (0, 1) = (–1, 0) = –1

Aside Consider $i(1 + 0i) = i$ geometrically:



Recall the rotation vector: $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{bmatrix}$

The new vector has the same magnitude as $\begin{bmatrix}x \\ y\end{bmatrix}$ but is separated by θ.

Let θ = 90, and let $1 + 0i = \begin{bmatrix}x \\ y\end{bmatrix}$, then

$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ 1\end{bmatrix} = 0 + 1i$.

So then $i = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$. And we can verify that $1 = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$.

Thus, $\forall\ a, b \in \mathrm{R}$, $a + bi = a \bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \bullet \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. And note that $(a+bi)(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \bullet \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix} = ac - bd + (ad + bd)i.$

So complex numbers can be represented by matrices with the usual matrix operations of addition and multiplication and scalar multiplication.

SO(3) is the special orthogonal group in 3 dimensions $\mathbb{R}^3$.

Definition $\mathbb{C}^n = \{(z_1, z_2, ..., z_n) |\ z_i \in \mathbb{C}, i = 1, n\}$

$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) |\ x_i \in \mathbb{R}, i = 1, n\}$

These are vector spaces with similar operations.

They both have the usual vector space operations of addition and scalar multiplication. These are called linear operations because the operation does not produce a result that lies outside the space you are in.

But the multiplication of vectors (not linear operation) differs between them. In $\mathbb{C}^n$ we have

$(w_1, w_2, ..., w_n) \bullet (z_1, z_2, ..., z_n) = w_1 \bar{z}_1 + w_2 \bar{z}_2 + \cdots + w_n \bar{z}_n$.

While in $\mathbb{R}^n$ we have

$(x_1, x_2, ..., x_n) \bullet (y_1, y_2, ..., y_n) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.