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Theorem Archimedean Property: Let $x \in \mathbb{R}$, $x > 0$. Then $\forall\, y \in \mathbb{R}\ \exists\, n \in \mathbb{N}$ such that $nx > y$.

Theorem Density of $\mathbb{Q}$ in $\mathbb{R}$. $\forall\, x, y \in \mathbb{R}$, $x < y$, $\exists\, p \in \mathbb{Q}$ such that $x < p < y$.

Theorem $\forall\, x \in \mathbb{R}$, $x > 0$, and $\forall\, n \in \mathbb{N}$, $\exists!\, y > 0$ such that $y^n = x$.

Homework 2. If we define the order $(a, b) \le (c, d)$ if $a \le c$ and $b \le d$ then the positive values lie in the first quadrant of the complex plane.

If you add 2 vectors, you use the parallelogram law to find the resultant which will also lie in the first quadrant if the 2 vectors are positive. But when you multiply 2 vectors, you add their directional angles, which may put the resultant vector in a quadrant that is no longer positive. Thus, the product of two vectors may not be positive.

3. $n = p_1^{n_1} \bullet p_2^{n_2} \bullet\bullet\bullet\, p_m^{n_m}$ where at least one exponent is odd.

Assume $\sqrt{n} = a/b$, then $b^2 n = a^2$. Thus

$$b_1^{2n_1} \bullet b_2^{2n_2} \bullet\bullet\bullet\, b_{m_1}^{2n_m} \bullet p_1^{n_1} \bullet p_2^{n_2} \bullet\bullet\bullet\, p_{m_2}^{n_{m2}} = a_1^{2n_1} \bullet a_2^{2n_2} \bullet\bullet\bullet\, a_{m_3}^{2n_{m3}}.$$

This gives us an odd power of a prime on the left and an even power of the same prime on the right.

4. $2^{p/q} = 3 \Rightarrow 2^q = 3^p$ which gives us 2 distinct prime factorizations of the same number, contrary to Fund. Theorem of Arithmetic.

Review Know how to construct the real numbers.

Know the definition of a cut.

$\mathbb{R} = \{\text{cuts in } \mathbb{Q}\}$

cut = $\alpha \subset \mathbb{Q}$ such that (1) $\alpha \ne \varnothing$; $\alpha \ne \mathbb{Q}$;

(2) $\forall\, p \in \alpha\ \forall\, p \in \mathbb{Q}$, $q < p \Rightarrow q \in \alpha$;

(3) $\forall\, p \in \alpha\ \exists\, r \in \alpha$ such that $p < r$

Theorem 1.20 (a) Archimedean Property
Let $x \in \mathbb{R}$, $x > 0$. Then $\forall\, y \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ such that $nx > y$.
(b) Density of $\mathbb{Q}$ in $\mathbb{R}$. (In topology we have $A \subset X$ is dense if $\overline{A} = X$)
$\forall\, x, y \in \mathbb{R}$, $x < y$, $\exists\, p \in \mathbb{Q}$ such that $x < p < y$.

Proof:
(a) Case 1: $y > 0$. Suppose the result is not true.

Then $\exists\, y \in \mathbb{R}^+$ such that $\forall\, n \in \mathbb{N}$ we have $nx \le y$.

Let $A = \{nx \mid n \in \mathbb{N}\}$. This is bounded above.

Use the least upper bound property of $\mathbb{R}$ to conclude $\alpha = \sup A$ exists.

So $\alpha - x$ is not an upper bound. So $\exists\, m \in \mathbb{N}$ such that $\alpha - x < mx$.

Thus $\alpha < mx + x = x(m + 1) \in A$. But this contradicts that $\alpha = \sup A$.

Case 2: If $y \le 0$, we choose $n = 1$ and have $1 \bullet x > y$ as $x > 0$.

(b) Case 1: $0 < x < y$.
By (a) $\exists\, n \in \mathrm{N}$ such that $n(y - x) > 1$, hence $1/n < y - x$.

Let $A = \{\frac{m}{n} | m \in \mathrm{N}$ and $\frac{m}{n} \le x\}$. This set is bounded above by x.

Then $\exists\, m_0 \in \mathrm{N}$ such that $m_0/n = \sup A$. And $m_0/n \in A$ as A is finite.

We know $(m_0 + 1)/n > x$ as $\sup A = m_0/n < (m_0+1)/n$,

so then $(m_0 + 1)/n \notin \mathrm{A}$, hence $(m_0 + 1)/n > x$.

So then $m_0/n \le x \Rightarrow (m_0 + 1)/n = m_0/n + 1/n < x + y - x = y$.

Thus $x < (m_0 + 1)/n < y$.

Case 2: If $x < 0 < y$, then $0 \in \mathrm{Q}$, and the result holds.

Case 3: If $x < y < 0$, then $0 < -y < -x$. So then by Case 1: $\exists\, p \in \mathrm{Q}$

such that $-y < p < -x$, thus $x < -p < y$.

Theorem 1.21

$\forall\, x \in R$, $x > 0$, and $\forall\, n \in N$, $\exists!\, y > 0$ such that $y^n = x$ (i.e. $y = \sqrt[n]{x}$).

Proof:

Uniqueness: Suppose $\exists\, y_1, y_2 \in R$ such that $0 < y_1 < y_2$.
We will show $y_1^n \neq y_2^n$ by induction.
Let $n = 1$, then $y_1^2 < y_1y_2$ and $y_1y_2 < y_2^2$, hence $y_1^2 < y_2^2$.
Let $n > 1$ and assume $y_1^n < y_2^n$.
Then $y_1^n\, y_1 < y_2^n\, y_1$ and $y_2^n\, y_1 < y_2^n\, y_2$. Thus $y_1^{n+1} < y_2^{n+1}$.
Let $x \in R^+$, let $n \in N$, and let $E = \{t \in R \mid t^n < x\}$.
If $t = x/(x + 1)$, then $(x/(x + 1))^n < x/(x + 1) < x$. So $E \neq \emptyset$.
If $t = x + 1$, then $(x + 1)^n > x + 1 > x$, so E is bounded above by $x + 1$.
Thus $y = \sup E$ exists.

Claim: $y^n = x$.
Suppose $y^n < x$.
We will find a contradiction by looking for $h > 0$ such that
$y + h$ is not an upper bound of E, (i.e. $(y + h)^n < x$).
We will use the identity

$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + b^0a^{n-1})$.
$\leq (b - a)(b^{n-1} + b^{n-2}b + \cdots + b^0b^{n-1})$ when $a < b$
$= (b - a)(b^{n-1} + b^{n-1} + \cdots + b^{n-1})$
$= (b - a)nb^{n-1}$.

Then for our contradiction, we work backwards from
$(y + h)^n < x$
$(y + h)^n - y^n < x - y^n$ so that we can use the identity above
$(y + h)^n - y^n < (y + h - y)n(y + h)^{n-1}$ by the identity since $y < y + h$
So then we want $hn(y + h)^{n-1} < x - y^n$.
This tells us that we want $h < (x - y^n)/[n(y + h)^{n-1}]$.
But we need this value in terms of y, x and n only,
so we can make $h < 1$.
Then let $h < \min \{1, (x - y^n)/[n(y + 1)^{n-1}]\}$.
Thus, $(y + h)^n - y^n < hn(y + 1)^{n-1} < x - y^n$, hence $(y + h)^n < x$,
the desired contradiction.
Now suppose $y^n > x$. We will work backwards again. We want
to find $k > 0$ such that $y - k$ is not an element of E.
$(y - k)^n > x$
$(y - k)^n - y^n > x - y^n$
$y^n - (y - k)^n < y^n - x$
$y^n - (y - k)^n < (y - (y - k))ny^{n-1} = kny^{n-1}$
So then we want $kny^{n-1} \leq y^n - x$, so we can let $k = (y^n - x)/(ny^{n-1})$.
Thus, $y^n - (y - k)^n < kny^{n-1} < y^n - x$, hence $(y - k)^n > x$ as desired.