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Homework Discussion

In Practice Problems #4, each problem is about $f: A \to B$ where $A_1, A_2 \subset A$ and $B_1\, B_2 \subset B$.
$f(A_1) = \{f(a) \in B \mid a \in A_1\}$
$f^{-1}(B) = \{a \in A \mid f(x) \in B\}$

1. (a) Prove $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
Proof: $\subseteq$: Let $b = f(A_1 \cup A_2)$.
Then $\exists\, a \in A_1 \cup A_2$ such that $f(a) = b$.
So $a \in A_1$ or $a \in A_2$. Hence $b \in f(A_1)$ or $f(a) \in f(A_2)$.
Thus $b \in f(A_1) \cup f(A_2)$. You complete the proof.

(d) Prove $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
Proof: $\supseteq$: Let $a \in f^{-1}(B_1) \cap f^{-1}(B_2)$, then $a \in f^{-1}(B_1)$ and $a \in f^{-1}(B_2)$.
So $f(a) \in B_1$ and $f(a) \in B_2$.
Thus $f(a) \in B_1 \cap B_2$. Hence $a \in f^{-1}(B_1 \cap B_2)$. You complete the proof.

2. (a) Prove if f is 1–1, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.
Proof: We have $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ always.
Let $b \in f(A_1) \cap f(A_2)$, then $b \in f(A_1)$ and $b \in f(A_2)$.
Thus, $\exists\, a_1 \in A_1$ and $a_2 \in A_2$ such that $f(a_1) = b = f(a_2)$.
Since f is 1–1, then $a_1 = a_2$.
Thus, $a_1 \in A_1 \cap A_2$, hence $b \in f(A_1 \cap A_2)$.

Counterexample if f is not 1–1: $f(x) = x^2$, $A_1 = \{-2\}$, $A_2 = \{2\}$.
Then $f(A_1 \cap A_2) = \varnothing \neq \{4\} = f(A_1) \cap f(A_2)$.

(c) Prove if f is 1–1, then $f^{-1}(f(A_1)) = A_1$ for every $A_1 \subset A$.
Proof: $A_1 \subset f^{-1}(f(A_1))$ always
$a \in A_1 \Rightarrow f(a) \in f(A_1) \Rightarrow a \in f^{-1}(f(A_1))$.
If $a \in f^{-1}(f(A_1))$, then $f(a) \in f(A_1)$.
So $\exists\, a_1 \in A_1$ such that $f(a_1) = f(a)$.
Since f is 1–1, then $a_1 = a$. Thus, $a \in A_1$.

If $f: A \to B$ is 1–1, then $\exists\, g: f(A) \to A$ such that $g \circ f(a) = a$, $\forall\, a \in A$.
This is a left inverse property.
We don't always have right inverses as $f \circ g(b) = f(g(b))$ but b might not be an element of $f(A)$.

4. (a) If $f: A \to B$ and $g: B \to C$ are 1-1 and onto functions, prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
Proof: $(g \circ f)^{-1}(C) = \{a \in A \mid (g \circ f)(a) \in C\}$.
Let $a \in (g \circ f)^{-1}(C)$. Then $(g \circ f)(a) = c$ for some $c \in C$.
So then $(g(f(a)) = c$. Thus $f(a) = g^{-1}(c)$, hence $a = (f^{-1} \circ g^{-1})(c)$.
$\therefore\ a \in f^{-1} \circ g^{-1}(C)$.

A set $\mathcal{F} = \{f: A \to A \mid f \text{ is a bijection}\}$ with composition, forms a non-commutative group.

Chapter 2

Definition **2.3** $A \sim B \Leftrightarrow \exists f: A \to B$ which is a bijection.
$[A] = \{\text{sets } B \mid A \sim B\}$.

Example In $\mathbb{R}^2$, define $(x, y) \sim (z, v)$ if $\exists\ \alpha > 0$ such that $(z, v) = \alpha(x, y)$.
Note that $\alpha = 0$ is a problem as we would lose reflexivity.
That is $(0, 0) \sim (2, 4)$ as $(0, 0) = 0(2, 4)$, but
$(2, 4) \sim (0, 0) \Rightarrow \exists\ \alpha$ such that $(2, 4) = \alpha(0, 0)$, impossible.
So $[(x, y)] = \{(z, v) \mid (z, v) = \alpha(x, y) \text{ for some } \alpha > 0\}$
This set lies on the line containing the origin and (x, y).

In general [A] is identified with the number of elements of A.
That is, card(A) = card($\cup\{\text{sets } B \mid A \sim B\}$).

Definition **2.4** Denote $\{1, 2, ..., n\} = \mathbb{N}_n$.
(a) A is finite if $\exists\ n \in \mathbb{N}$ such that $A \sim \mathbb{N}_n$, or $A = \varnothing$.
(b) A is infinite if it is not finite.
(c) A is countable if $A \sim \mathbb{N}$.
(d) A is uncountable if it is not at most countable.
(e) A is at most countable if A is finite or countable.

Examples $\mathbb{N}$ is countable as $\mathbb{N} \sim \mathbb{N}$.
$\mathbb{Z}$ is countable as we have $f: \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ -\dfrac{n-2}{2} & \text{if } n \text{ is odd} \end{cases}$$

We have seen how to order $\mathbb{Q}_+$.

This is a countable union of countable sets.

To show $\mathbb{Q}$ is countable, we note that $\mathbb{Q} = \mathbb{Q}_- \cup \{0\} \cup \mathbb{Q}_+$.
This is a finite union of countable sets.

Question What step do we take to go from countable to uncountable?

Definition Define the power set of X to be the set of all subsets of X, denoted $\mathscr{P}(X)$.

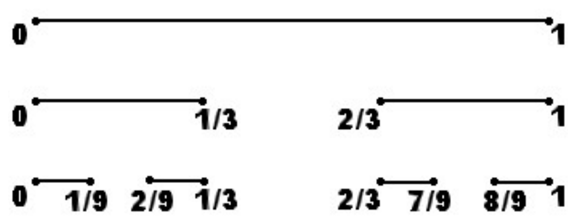
Example $X \nsim \mathscr{P}(X)$ as $\mathscr{P}(X)$ is larger than X.
So $\mathbb{Q} \nsim \mathscr{P}(\mathbb{Q})$.
$\mathscr{P}(\mathbb{Q}) \sim \mathbb{R}$.
We call the cardinality of $\mathbb{Q}$, $\aleph_0$ and the cardinality of $\mathbb{R}$, $\mathfrak{c}$

Continuum Hypothesis: There is no number between $\aleph_0$ and $\mathfrak{c}$.

The axiom that given a line and a point not on the line, there is exactly one line through that point that is parallel to the given line is similar to the continuum hypothesis.
We accept it, but cannot prove it.

Examples $\mathbb{R}$ is uncountable.
$(0, 1)$ is uncountable.
$\mathbb{R}^n$ is uncountable.

The Cantor Set is uncountable.



Notice that we take out
$1/3 + 2/9 + 4/27 + 8/81\cdots$
$= 1/3(1 + 2/3 + 4/9 + 8/27\cdots)$
$= \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$. The remaining length $= 1 - \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1/(1-2/3) = 1$.
Although the endpoints remain, which are countable, there are other numbers that remain also. We can write each number of the Cantor Set using base 3. That is using the digits 0, 1, and 2, and the place value system $\cdots 3^3, 3^2, 3^1, 3^0, 3^{-1}, 3^{-2}, \cdots$,
numbers in the Cantor Set look like 0.20220200... .
Numbers of this form are uncountable as stated in the following theorem (substitute 2 for 1).

Theorem 2.14 The set all all sequences whose elements are the digits 0 and 1 is uncountable.