Content:
Review for exam
Theorem Cauchy-Schwarz $|a \bullet b|^2 \leq |a|^2 \bullet |b|^2$.
Theorem Let $f : A \to B$ be a 1-1 function. Then f has a left inverse $g : B \to A$ such that $g \circ f(a) = a \; \forall \; a \in A$.
Definitions cardinality "≤".

Review For exam, review up to top of p. 26.

Know definitions for order axioms, upper/lower bounds,

least upper/greatest lower bounds, least upper bound propert.

Know how to prove $\sqrt{7}$ is irrational.

Know examples that have certain properties and those that don't.

Know Theorem 1.11, p. 5.

Won't ask about axioms of a field (but must be able to use them).

Know 1.17, connection between order and field axioms.

Know Proposition 1.18.

From Theorem 1.19, know how to construct $\mathbb{R}$ from $\mathbb{Q}$.

Know definition of a cut. Could ask about Step 3.

Know Theorem 1.20 (Archimedean Property of the Reals and

Density of Rationals in the Reals).

Theorem 1.21, Know how to define the nth root. It's the sup of a set.

For Complex Numbers, know Theorem 1.31, 1.33, 1.35.

For Reals, know Theorem 1.36, 1.37 (properties a, b, c, e).

Be able to show the parallogram law fails if p ≠ 2.

(Hint: Let a = (1, 0), b = (0, 1)).

Know the Young, Hölder, Minkowski inequalities.

Theorem Cauchy-Schwarz

$|a \bullet b|^2 \leq |a|^2 \bullet |b|^2$ where $a \bullet b = \sum_{i=1}^{n} a_i \overline{b}_i$.

Proof:

$0 \leq |a + tb|^2$ where $t \in \mathbb{R}$.

$= (a + tb)(a + tb)$

$= a \bullet a + a \bullet tb + tb \bullet a + t^2 b \bullet b$

$= |a|^2 + t(a \bullet b + b \bullet a) + t^2|b|^2$

$= |a|^2 + 2t(\text{Re}(a \bullet b)) + t^2|b|^2$

Discriminant: $4\text{Re}(a \bullet b)) - 4|a|^2 \bullet |b|^2 \leq 0$

$\text{Re}(a \bullet b)^2 \leq |a|^2 \bullet |b|^2$

?????

Alternative proof:

Let $A = |a|^2$, let $B = |b|^2$, and let $C = a \bullet b$.

Assume $B > 0$, as result is trivially true for $B = 0$.

$\sum_{j=1}^{n} \left| |b|^2 a_j - (a \bullet b) b_j \right|^2 = \sum_{j=1}^{n} \left(|b|^2 a_j - (a \bullet b) b_j \right)\left(|b|^2 \overline{a}_j - (\overline{a \bullet b}) \overline{b}_j \right)$

$= |b|^4 \sum_{j=1}^{n} a_j \overline{a}_j - |b|^2 (\overline{a \bullet b}) \sum_{j=1}^{n} a_j \overline{b}_j - |b|^2 (a \bullet b) \sum_{j=1}^{n} \overline{a}_j b_j + |a \bullet b|^2 |b|^2$

$= |b|^4 |a|^2 - |b|^2 (\overline{a \bullet b})(a \bullet b) - |b|^2 (a \bullet b)(\overline{a \bullet b}) + |a \bullet b|^2 |b|^2$

$= |b|^4 |a|^2 - |b|^2 \left((\overline{a \bullet b})(a \bullet b) - (a \bullet b)(\overline{a \bullet b}) \right) + |a \bullet b|^2 |b|^2$

$= |b|^4 |a|^2 - |b|^2 |a \bullet b|^2 - |b|^2 |a \bullet b|^2 + |a \bullet b|^2 |b|^2$

$= |b|^2 \left(|b|^2 |a|^2 - |a \bullet b|^2 \right)$.

Since $B = |b|^2 > 0$, then $|b|^2|a|^2 - |a \bullet b|^2 > 0$, hence $|a \bullet b|^2 < |b|^2|a|^2$.

Theorem Let $f : A \to B$ be a 1-1 function. Then f has a left inverse $g : B \to A$ such that $g \circ f(a) = a \ \forall \ a \in A$.

Definition $A \sim B \Leftrightarrow \exists f : A \to B$ that is bijective.

card A = card $B \Leftrightarrow A \sim B$.

card $A \leq$ card $B \Leftrightarrow \exists f : A \to B$ that is 1-1.

card $B \geq$ card $A \Leftrightarrow \exists g : B \to A$ that is onto.

We can show "≤" is an equivalence relation.

(i) card $A \leq$ card A by $f(x) = x$.

(ii) card $A \leq$ card B and card $B \leq$ card $A \Rightarrow$ card A = card B.

card $A \leq$ card $B \Rightarrow \exists f : A \to B$ that is 1-1.

card $B \leq$ card $A \Rightarrow \exists g : B \to A$ that is 1-1.

We can define $h : A \to B$ that is 1-1 and onto.

Example $(0, 1) \sim [0, 1)$