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Chapter 1, The Real and Complex Number System

We can construct the real number system, but first we must construct the following:
(1) Natural Numbers = {1, 2, 3, ...}. These can be constructed inductively. This is a well-ordered set.
(2) Integers = $(\mathbb{Z}, +)$. Just include the opposites of all natural numbers + 0.
(3) Rational Numbers, $(\mathbb{Q}, +)$. $\mathbb{Q}^* = \mathbb{Q} - \{0\}$. $(\mathbb{Q}, +, \bullet)$ is a field.

Example If $x^2 = 2$, then this equation has no solution in the field $\mathbb{Q}$.

Proof:

Suppose $\exists\, p = \dfrac{n}{m} \in \mathbb{Q}$ such that $p^2 = 2$ where $(n, m) = 1$.

Then $\dfrac{n^2}{m^2} = 2$.

$\Rightarrow n^2 = 2m^2$,

$\Rightarrow n^2$ is even,

$\Rightarrow n$ is even by the Fundamental Thm of Arithmetic,

$\Rightarrow n = 2k$, for some integer k,

$\Rightarrow (2k)^2 = m^2$,

$\Rightarrow m^2$ is even,

$\Rightarrow m$ is even by the Fundamental Thm of Arithmetic,

$\Rightarrow$ $(n, m) \geq 2$, contrary to our assumption that $(n, m) = 1$.

$\therefore$ x^2 has no solution in the field $\mathbb{Q}$.

Example If $A = \{p \in \mathbb{Q}: p^2 < 2\}$, then $\forall\, p \in A$, $\exists\, q \in A$ such that $p < q$.

Then $p < q = \dfrac{2p+2}{p+2} = p - \dfrac{p^2-2}{p+2} < \sqrt{2}$.

We can verify $p < \dfrac{2p+2}{p+2}$ by noting that $p > 0$ and $p^2 < 2$

$\Rightarrow p^2 + 2p < 2 + 2p \Rightarrow p(p + 2) < 2p + 2 \Rightarrow p < \dfrac{2p+2}{p+2}$.

(See http://math.berkeley.edu/~gbergman/ug.hndts/m104_Rudin_exs.pdf for an explanation of how Rudin obtained this formula.)

Definition 1.3 Element, Subset, Proper Subset

$A \neq \varnothing$ (A is nonempty.)

$x \in A$, $x \notin A$ (x is/is not an element of A)

$A \subset B$ ($\forall\, x \in A$, $x \in B$), or $x \in A \Rightarrow x \in B$.

$A = B \Leftrightarrow A \subset B$ and $B \subset A$.

Definition 1.5 Order

Let $S \neq \varnothing$. Define "<" on S having the following properties:

(i) If $x, y \in S$, then exactly one of the following are true

$x < y$; $x = y$; or $y < x$.

(ii) If $x, y, z \in S$ where $x < y$ and $y < z$, then $x < z$.

(We could add $x \leq y \Leftrightarrow x < y$ or $x = y$, the negation of $x > y$.)

Define "≤" on S have the following properties:

(i) $\forall\, x \in S$, $x \leq x$.

(ii) If $x, y \in S$ where $x \leq y$ and $y \leq x$, then $x = y$.

(iii) If $x, y, z \in S$ where $x \leq y$ and $y \leq z$, then $x \leq z$.

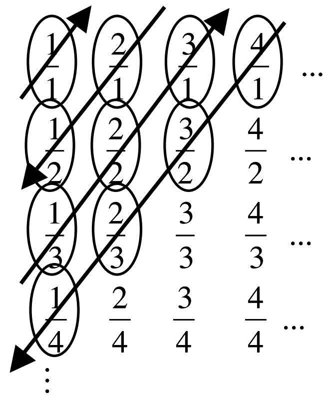
Note The definition "≤" does not necessarily allow two arbitrary elements to be compared. We call this a partial ordering.

Example $S = \mathbb{R}^2$. Define "≤" on S by $(x_1, x_2) \leq (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$.

This order obeys the conditions for "≤", but (1, 3) and (0, 4) cannot be compared.

Note In a totally ordered set, we must be able to compare any two elements.

Example Every set can be well-ordered (Zermello's Theorem).
$\mathbb{N}$ is ordered already.
Z can be ordered as follows: Z = {0, 1, −1, 2, −2, ...}.
Q can be orderd by arranging the elements and ordering them as follows:



$\mathbb{R}$ can be ordered. However, this requires the Axiom of Choice,
$\forall$ collection of sets A_i, $i \in I$, $\exists$ a function $f: I \to \cup A_i$ such that $f(i) \in A_i$.

Definition 1.7 Bounded Above/Below, Upper/Lower Bound
Let $S \neq \varnothing$ be an ordered set.
Consider $E \subset S$. E is bounded above if $\exists\ \beta \in S$ such that $x \leq \beta\ \forall\ x \in E$.
β is called an upper bound of E.
And E is bounde below if $\exists\ \alpha \in S$ such that $\alpha \leq x\ \forall\ x \in E$.
α is called a lower bound of E.

Definition 1.8 Supremum, Infimum
β is the least upper bound, or supremum of $E \subset S$, denoted sup(E) if
(i) β is an upper bound, and
(ii) If γ is another upper bound, then $\beta < \gamma$.
(In Rudin Text: (ii) If $\gamma < \beta$, then γ is not an upper bound.)

Note If $S = \mathbb{R}$, then (ii) can be written as $\forall\ \varepsilon > 0, \exists\ \gamma \in E$ such that $\beta - \varepsilon < \gamma \leq \beta$.

α is the greatest lower bound, or infimum of E, denoted inf(E) if
(i) α is a lower bound, and
(ii) If γ is another lower bound, then $\gamma < \alpha$.

Example If $S = \mathbb{Q}$ and $A = \{p \in \mathbb{Q}: p^2 < 2\}$ then
(i) A is bounded above, but
(ii) A does not have a least upper bound in S.

Definition 1.10 Least Upper/Greatest Lower Bound Property.
S has the least upper bound property if
whenever $E \subset S$, $E \neq \emptyset$, and E is bounded above, $\sup(E)$ exists.
S has the greatest lower bound property if
whenever $E \subset S$, $E \neq \emptyset$, and E is bounded below, $\inf(E)$ exists.

Theorem 1.11 Suppose that S has the least upper bound property,
then S also has the greatest lower bound property.
That is, if $B \subset S$, $B \neq \emptyset$, and B is bounded below, then $\inf(B)$ exists.

Proof:
Let $B \subset S$ such that $B \neq \emptyset$ and B is bounded below.
Let L be the set of lower bounds of B. By assumption, $L \neq \emptyset$.
Let $y \in L$. Then $\forall\, x \in B$, $y \leq x$. Hence L is bounded above.
In particular, $\forall\, x \in B$, x is an upper bound of L.
Since S has the least upper bound property, then $\alpha = \sup(L)$ exists in S.
We only need to show $\alpha = \inf(B)$.
Note that by definition of supremum, $\forall\, \gamma < \alpha$, γ is not an upper bound.
So then we have
(i) $\alpha \leq x\ \forall\, x \in B$; and
(ii) If $\alpha < \gamma$, $\gamma \notin L$, or equivalently γ is not a lower bound of B.
$\therefore$ $\inf(B) = \alpha$, hence S has the greatest lower bound property.

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