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Definition	Ordered field
Theorem	1.19 There exists an ordered field $\mathbb{R}$ which has the lub property and which contains $\mathbb{Q}$ as a subfield.

Homework Ch.1 #1 $r \in \mathbb{Q}$, $r \ne 0$, $x \notin \mathbb{Q}$. Prove $r + x$, $rx \notin \mathbb{Q}$.
If $r + x \in \mathbb{Q}$, then $r + x = a/b$ for some $a, b \in \mathbb{Z}$, $b \ne 0$.
So then $x = a/b - r \in \mathbb{Q}$, a contradiction to $x \notin \mathbb{Q}$.

Ch.1 #2 $r \in \mathbb{Q}$. $r^2 \in \mathbb{Q}$?
$r^2 = 4 \bullet 3 \Rightarrow m^2/n^2 = 4 \bullet 3 \Rightarrow m^2 = 4 \bullet 3n^2 \Rightarrow m = 3k \Rightarrow 9k^2 = 4 \bullet 3n^2 \Rightarrow 3k^2 = 4n^2 \Rightarrow n = 3j$.

If we know $\sqrt{12}$ exists, then we can use #1. $\sqrt{12} = 2\sqrt{3}$.
If we don't know exists, then to prove there is no $r \in \mathbb{Q}$ such that $r^2 = 12$ requires a different approach.

Review If S is a nonempty set, then we can define "<".
With $(S, <)$ we can define the lub property.
If every nonempty subset $E \subset S$ that is bounded above has a least upper bound, then S has the least upper bound property.
This is also called the "Completeness Property" or "Completeness Axiom".

Example Let $S = \mathbb{Z}$. Let $E \subset \mathbb{Z}$ such that $E \ne \varnothing$ and E is bounded above.
Then $\exists\, n_0 \in \mathbb{Z}$ such that $a \le n_0\ \forall\, a \in E$.

Proof:
Let n_0 be an upper bound of E.
Case 1: If $n_0 = \sup E$, then we're done.
Case 2: If $n_0 \ne \sup E$, then $n_0 - 1$ is an upper bound of E.
  Case 1: If $n_0 - 1 = \sup E$, then we're done.
  Case 2: If $n_0 - 1 \ne \sup E$, then consider $n_0 - 2$.
As $E \ne \varnothing$, this process must end.

Example $\mathbb{Q}$ does not have the lub property as $\sup E \notin \mathbb{Q}$ where $E = \{r \in \mathbb{Q} : r^2 < 2\}$.

Definition 1.12 A *field* is a set F with 2 operations, + and • which satisfy the field axioms

"+" $\forall\, x, y, z \in F$, we have
closure: $x + y \in F$
commutativity: $x + y = y + x$
associativity: $x + (y + z) = (x + y) + z$
identity element: $\exists\, 0 \in F$ such that $x + 0 = 0 + x = x$
inverse element: $\exists\, -x$ such that $x + -x = -x + x = 0$

"•" $\forall\, x, y, z \in F$, we have
closure: $xy \in F$
commutativity: $xy = yx$
associativity: $x(yz) = (xy)z$
identity: $\exists 1 \in F$ such that $1 \bullet x = x \bullet 1 = x$
inverse: $\forall\, x \neq 0, \exists\, 1/x$ such that $x \bullet 1/x = (1/x) \bullet x = 1$

Distributive law: $x(y + z) = xy + xz$

Definition 1.17 An *ordered field* is a field F which is also an ordered set, such that
(i) If $x, y, z \in F$ and $y < z$, then $x + y < x + z$
(ii) If $x, y, z \in F$, $y < z$ and $0 < x$, then $xy < xz$
(Book's (ii)') If $x, z \in F$, $x > 0$, and $z > 0$, then $xz > 0$.

(ii) $\Rightarrow$ (ii)' If $y = 0$ and $z > y$, hence $z > 0$, then $z \bullet x > 0 \bullet x = 0$.
(ii)' $\Rightarrow$ (ii) If $0 < z - y$ and $0 < x$, then $0 < x(z - y)$, and so $xy < xz$.

Propositions 1.14, 1.15, 1.16, 1.18 check these, but they won't be exam material.

Theorem 1.19 There exists an ordered field $\mathbb{R}$ which has the lub property and which contains $\mathbb{Q}$ as a subfield.

Step 1 Start with $\mathbb{Q}$ and build $\mathbb{R}$.
Define "cuts" in $\mathbb{Q}$ as $\alpha \subset \mathbb{Q}$ where
(i) $\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$
(ii) If $p \in \alpha$ and $q \in \mathbb{Q}$, then $q < p \Rightarrow q \in \alpha$
(iii) If $p \in \alpha$, then $\exists\, r \in \mathbb{Q}$ such that $p < r$
$\mathbb{R} = \{\alpha |\ \alpha \text{ is a cut in } \mathbb{Q}\}$

Step 2 Define $\alpha < \beta$ if $\alpha \subset \beta$ and $\alpha \neq \beta$.
If $\alpha, \beta, \gamma \in \mathbb{R}$, then
(i) $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$; and
(ii) If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$
Now we have $(\mathbb{R}, <)$, an ordered set.

Step 3 Let E be a nonempty subset of $\mathbb{R}$. Assume β is an upper bound of E.
Define $\gamma = \cup_{\alpha \in E} \alpha$.
Since $E \neq \emptyset$, $\exists\ \alpha_0 \in E$ such that $\alpha_0 \neq \emptyset$. Since $\alpha_0 \subset \gamma$, then $\gamma \neq \emptyset$.
$\gamma \subset \beta$, so $\gamma \neq \mathbb{Q}$. If $p \in \gamma$, then $p \in \alpha_1$ for some $\alpha_1 \in E$.
If $q < p$, then $q \in \alpha_1$, hence $q \in \gamma$.
And if $r \in \alpha_1$ such that $r > p$, then $r \in \gamma$ since $\alpha_1 \subset \gamma$.
Thus $\gamma \in \mathbb{R}$.
$\alpha \leq \gamma$ for every $\alpha \in E$.
If $\delta < \gamma$, then $\exists\ s \in \gamma$ such that $s \notin \delta$. Since $s \in \gamma$, $s \in \alpha$ for some $\alpha \in E$.
Hence, $\delta < \alpha$, and δ is not an upper bound of E.
$\therefore\ \gamma = \sup E$.
Thus, $\mathbb{R}$ has the lub property.

Step 4 If $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, we define $\alpha + \beta$ to be the set of all sums $r + s$, where $r \in \alpha$ and $s \in \beta$.
Verify the 5 axioms for "+".

Step 5 If α, β, and $\gamma \in \mathbb{R}$, and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

Step 6 If $\alpha, \beta \in \mathbb{R}^+$, define $\alpha\beta = \{p \mid p \leq rs \text{ for } r \in \alpha, s \in \beta \text{ where } r > 0, s > 0\}$

Step 7 Set $\alpha 0^* = 0^* \alpha = 0^*$.
Define
$$\begin{aligned} \alpha\beta &= (-\alpha)(-\beta) \text{ if } \alpha < 0^*, \beta < 0^* \\ &= -[(-\alpha)\beta] \text{ if } \alpha < 0^*, \beta > 0^* \\ &= -[\alpha \bullet (-\beta)] \text{ if } \alpha > 0^*, \beta < 0^* \end{aligned}$$

Step 8 Identify each $r \in \mathbb{Q}$ with $r^* = \{q \in \mathbb{Q} \mid q < r\}$
These cuts satisfy the following relations:
(a) $r^* + s^* = (r + s)^*$
(b) $r^*s^* = (rs)^*$
(c) $r^* < s^*$ if and only if $r < s$.

Step 9 $\mathbb{Q} \subset \mathbb{R}$.