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Theorem **2.8** Every infinite subset of a countable set is countable.

Proof:
Let A be a countable set. Let E be an infinite subset of A.
A is countable $\Rightarrow A = \{x_1, x_2, x_3, ...\}$.
Define $f : \mathbb{N} \to E$ as follows:
If $x_1 \in E$, then $f(1) = x_1$ and then we check if $x_2 \in E$.
 If $x_2 \in E$, then $f(2) = x_2$ and then we check if $x_3 \in E$...
If $x_1 \notin E$, then we check if $x_2 \in E$.
If $x_2 \in E$, then $f(1) = x_2$ and then we check if $x_3 \in E$.
 If $x_3 \in E$, then $f(2) = x_3$ and then we check if $x_4 \in E$...
We can index elements of E as follows:
Let n_1 be the smallest possible integer such that $x_{n_1} \in E$.
Let n_k be the smallest possible integer greater than n_{k-1} such that $x_{n_k} \in E$.
So then, in general, $f(k) = x_{n_k}$. And we can describe E as $\{x_{n_k}\}_{k \in \mathbb{N}}$.
That is, $E = \{x_{n_1}, x_{n_2}, ... , x_{n_k}, ...\}$.
Thus $E \sim \mathbb{N}$, hence E is countable.

Proposition If A and B are countable, then $A \cup B$ is countable.

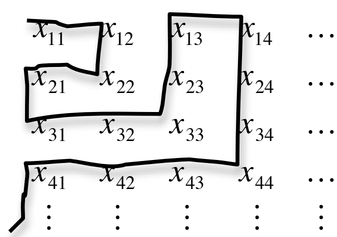
Proof:
Let $A = \{a_1, a_2, a_3, ...\}$, let $B = \{b_1, b_2, b_3, ...\}$, and let $C = \{a_1, b_1, a_2, b_2, ...\}$
Define $f : \mathbb{N} \to C$ by $f(x) = \begin{cases} a_k \text{ if } x = 2k - 1 (k \text{ a non-negative integer}) \\ b_k \text{ if } x = 2k \end{cases}$.
This is 1-1 and onto. Since $A \cup B \subseteq C$, then we can apply Theorem 2.8 to get that $A \cup B$ is countable.

Proposition If A is uncountable and $A \subset B$, then B is uncountable.

Proof:
Suppose B is countable. Since $A \subset B$, then by Theorem 2.8, A is countable. But this is contrary to our hypothesis that A is uncountable. $\therefore$ B is uncountable.

Theorem **2.12** If $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of countable sets. Then $\cup_{n\in\mathbb{N}} E_n$ is countable.

Proof:
For any n, $E_n = \{x_{n_k}\}_{k\in\mathbb{N}} = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\}$.
So then we can arrange the terms of $\cup_{n\in\mathbb{N}} E_n$ in the following matrix:



And we can write the elements in the sequence
$\{x_{11}, x_{12}, x_{22}, x_{21}, x_{32}, x_{33}, x_{23}, x_{13}, x_{14}, x_{24}, x_{34}, x_{44}, x_{43}, x_{42}, x_{41}, \ldots\}$

Conclusion: $\mathbb{Q}$ is countable. $\mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$ is a union of countable sets.

Corollary Suppose $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of at most countable sets.
Then $\cup_{n\in\mathbb{N}} E_n$ is at most countable.

Theorem **2.13** Let A be a countable set and $B = A \times A \times \cdots \times A$ (n times).
Then B is countable.

Proof:
$A = \{a_i\}_{i\in\mathbb{N}}$. So $A \times A = \{(a_i, a_j) | \, a_i, a_j \in A\}$.
Let $k \in \mathbb{N}$, and let $E_k = \{(a_k, a_j) | a_j \in A\}$.
Then $A^2 = \cup_{k\in\mathbb{N}} E_n$, which by Theorem 2.12 is countable.
Let $n > 2$ and assume A^n is countable.
Then $A^n = \{c_1, c_2, c_3, \ldots\}$.
Then $A^n \times A = \cup_{k\in\mathbb{N}} E_n$ where $E_n = \{(c_n, a_j) | c_n \in A_n \text{ and } a_j \in A\}$.
$\therefore$ A^{n+1} is countable. Hence B is countable.

Theorem **2.14** Let A be the set of all sequences whose elements are the digits 0 and 1. Then A is uncountable.

Proof:
Suppose A is countable. Then $A = \{s_1, s_2, s_3, ...\}$ where $s_n = \{s_{n_k}\}_{k \in \mathbb{N}}$.
For example, our first few sequences may look like

	s_{n_1}	s_{n_2}	s_{n_3}	s_{n_4}	s_{n_5}	s_{n_6}	s_{n_7}	
s_1 = (	0	1	1	0	0	0	1	...)
s_2 = (	1	1	0	1	0	1	0	...)
s_3 = (	0	0	1	0	0	0	0	...)

We can construct a new sequence $(x_n)_{n \in \mathbb{N}}$ of 0's and 1's as follows:

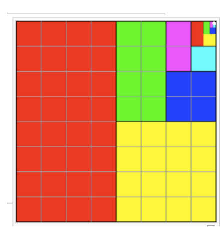
Let $n \in \mathbb{N}$, then $x_n = \begin{cases} 1 \text{ if } s_{n_n} = 0 \\ 0 \text{ if } s_{n_n} = 1 \end{cases}$.

In the example above, $x_{n_1} = 1$, $x_{n_2} = 0$, and $x_{n_3} = 0$.
So, in general,
$x_n \neq s_1$ as $x_{n_1} \neq s_{1_1}$, $x_n \neq s_2$ as $x_{n_2} \neq s_{2_2}$, ..., $x_n \neq s_n$ as $xx_{n_n} \neq s_{n_n}$, ...
Thus x_n differs from every sequence in A, hence $x_n \notin A$.
This is impossible as x_n is a sequence of 0's and 1's, and A is the set containing all such sequences.
$\therefore$ A is uncountable.

Corollary [0, 1] is uncountable.

Proof:
Let $x \in [0, 1]$. Then we can write x using base 2 notation.
Say, for example, $x = 0.1010010001... = 1/2 + 0/2^2 + 1/2^3 + \cdots$
Then x can be written as (1010010001...), a sequence of 1's and 0's.
And if y is any sequence of 1's and 0's, say (11011011011...)
then we can write y as $0.11011011011... \in [0, 1]$.
Note that 0.111111... = 1 as $\sum \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$

The image to the right
represents this geometrically.



Note It is not possible to construct a continuous function from [0, 1] to $\mathbb{R}$ as [0, 1] is compact, $\mathbb{R}$ is not compact, and continuous functions preserve compactness of sets.

We also have (0, 1) is uncountable as

$$(0, 1) \xrightarrow{x - \frac{1}{2}} \left(-\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{\pi x} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\tan x} \mathbb{R}.$$

Lemma Suppose $B \subset A$ and $\exists\, f: A \to B$ which is 1-1. Then $A \sim B$.

Proof:
Let $C_0 = A \backslash B$.
Assume $A \backslash B \neq \varnothing$ as $A \backslash B = \varnothing \Rightarrow A = B$ and there is nothing to prove.
Let $C_1 = f(C_0)$, ..., let $C_{n+1} = f(C_n)$, and let $C = \cup_{n \in \mathbb{N}} C_n$.
Define $h: A \to B$ by $h(z) = \begin{cases} f(z) \text{ if } z \in C \\ z \text{ if } z \notin C \end{cases}$.
To show h is 1-1, let $z_1, z_2 \in A$ such that $h(z_1) = h(z_2)$.
We have 3 cases:
(i) $h(z_1) = f(z_1) = f(z_2) = h(z_2)$. Then $z_1 = z_2$ as f is 1-1.
(ii) $h(z_1) = z_1 = f(z_2) = h(z_2)$. This is impossible as
$h(z_1) = z_1 \Rightarrow z_1 \notin C \Rightarrow f(z_2) \notin C \Rightarrow z_2 \notin C \Rightarrow h(z_2) \neq f(z_2)$.
(iii) $h(z_1) = z_1 = z_2 = h(z_2)$.
$\therefore$ h is 1-1.
To show h is onto, let $b \in B$. Then $b \in A$ as $B \subset A$.
If $b \in C$, then $\exists\, n \in \mathbb{N}$ such that $b \in C_n$.
Thus, $\exists\, a \in C_{n-1} \subset A$ such that $f(a) = b$, hence $h(a) = b$.
If $b \notin C$, then $h(b) = b$. $\therefore$ h is onto.
$\therefore A \sim B$.

Theorem **Schröder-Bernstein Theorem**
If card $A \leq$ card B and card $B \leq$ card A, then card A = card B.

Proof:
$\exists\, f: A \to B$ that is 1-1 as card $A \leq$ card B, and
$\exists\, g: B \to A$ that is 1-1 as card $B \leq$ card A. So then
$g: B \to g(B)$ is bijective. Thus
$g \circ f: A \to g(B)$ is 1-1. Note that $g(B) \subset A$, so by the lemma above,
$\exists\, k: A \to g(B)$ that is biject. And now we can define
$h: A \to B$ by $h(z) = g^{-1} \circ k(z)$.

To show h is 1-1, let $z_1, z_2 \in A$ such that $h(z_1) = h(z_2)$.
Then $g^{-1} \circ k(z_1) = g^{-1} \circ k(z_2)$. Since $k(z_1), k(z_2) \in g(B)$ and g is bijective, then $k(z_1) = k(z_2)$. Since k is bijective, then $z_1 = z_2$.

To show h is onto, let $b \in B$. Then $g(b) \in g(B)$ and since k is onto, then $\exists\, x \in A$ such that $k(x) = g(b)$.
And, $h(x) = g^{-1}(k(x)) = g^{-1}(g(b)) = b$ as g is 1-1.

$\therefore$ card A = card B.

Example Let $C_0 = \mathbb{Z} \setminus \mathbb{N}$.

Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(z) = \begin{cases} -4k - 3 \text{ if } z \in C_0 \\ 4k + 3 \text{ if } z \notin C_0 \end{cases}$.

Notice that $f(\mathbb{Z}) \cap \{\text{even integers}\} = \emptyset$, so $f(\mathbb{Z}) \neq \mathbb{Z}$.

Let $C_1 = f(C_0)$, ..., $C_{n+1} = f(C_n)$, $C = \cup\, C_n$.

Note that $C_1 = \{-3, 1, 5, 9, ...\}$

$C_1 = \{9, 7, 23, 39, ...\}$

$C_2 = \{39, 31, ...\}$

Define $h : \mathbb{Z} \to \mathbb{N}$ by $h(z) = \begin{cases} f(z) \text{ if } z \in C \\ z \text{ if } z \notin C \end{cases}$.

This function is 1-1 and onto by preceding lemma.