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Note $\mathscr{P}(A)$ = {all the subsets of A}.

Remark card $A = n \Rightarrow$ card $\mathscr{P}(A) = 2^n$.

Note $\mathscr{P}(A) = \{\varnothing, \{a_1\}, \{a_2\}, \{a_3\}, ...\}$ so single elements sets in $\mathscr{P}(A)$ are equal in number to the number of elements of A.

Note $A \subseteq B \Rightarrow$ card $A \leq$ card $B \Rightarrow \exists f: A \to B$ that is 1-1, or equivalently, $\exists g: B \to A$ that is onto.

Theorem card A < card $\mathscr{P}(A)$.

Proof:

Suppose card $A \geq$ card $\mathscr{P}(A)$. Then $\exists f: A \to \mathscr{P}(A)$ that is onto.

Consider $A_1 = \{a \in A : a \notin f(a)\}$. Note that $A_1 \subseteq A$, hence $A_1 \in \mathscr{P}(A)$.

Since f is onto and $A_1 \in \mathscr{P}(A)$, then $\exists a_1 \in A$ such that $f(a_1) = A_1$.

Since $A_1 \subseteq A$, then $a_1 \in A_1$, or $a_1 \notin A_1$.

If $a_1 \in A_1$, then $a_1 \notin f(a_1)$, but $f(a_1) = A_1$, so $a_1 \notin A_1$. $\Rightarrow\Leftarrow$

If $a_1 \notin A_1$, then $a_1 \in f(a_1) = A_1$. $\Rightarrow\Leftarrow$

$\therefore$ f is not onto. $\therefore$ card $A \ngeq$ card $\mathscr{P}(A)$, hence card A < card $\mathscr{P}(A)$.

Note card $\mathbb{N} = \aleph_0$, card $\mathbb{R} = \mathfrak{C}_1$, card $\mathscr{P}(\mathbb{R}) = \mathfrak{C}_2$.

Metric Spaces

Definition **2.15** Consider a non-empty set X. A function $d: X \times X \to \mathbb{R}$ is called a *metric* if

Positivity (a) $d(x, y) \geq 0\ \forall\ x, y \in X$; $d(x, y) = 0 \Leftrightarrow x = y$.

Symmetry (b) $d(x, y) = d(y, x)\ \forall\ x, y \in X$.

Triangle Inequaltiy (c) $d(x, y) \leq d(x, z) + d(z, y)\ \forall\ x, y, z \in X$.

Note If we replace $d(x, y) = 0 \Leftrightarrow x = y$ with $d(x, x) = 0\ \forall\ x \in X$, we have a semi-metric.

Example In $\mathbb{R}^2$, define $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|$. This is a semi-metric.

But we can fix this be defining equivalence classes.

Define $x \sim y \Leftrightarrow d(x, y) = 0$.

This is an equivalence relation: $x \sim x$.

$x \sim y \Rightarrow y \sim x$.

$x \sim y$ and $y \sim z \Rightarrow x \sim z$.

So then $x/\sim\ = \{[x] : x \in X\}$.

And this gives us $d([x], [y]) = 0 \Leftrightarrow [x] = [y]$.

Question How does the norm relate to the metric?
Every norm defines $d(x, y) = |x - y|$.
So *every norm space is also a metric space.*
Is the reverse true?
No. Metric spaces don't require any algebraic structure on the space, yet norm spaces do. A metric space can be simply a set of points with measurable distances between them. But recall that $|\mathbf{x}|_p = (\sum |x_i|^p)^{1/p}$.
So *not all metric spaces are norm spaces.*

Definition **2.18**
Let (X, d) be a metric space.
(a) A *neighborhood* (ball) centered at a point x of positive radius $r \in \mathbb{R}$, denoted $N_r(x) = \{y \in X \mid d(x, y) < r\}$.
(b) $x_0 \in X$ is a *limit point* of E if $\forall\, r > 0$, $N_r(x_0) \cap \{E \setminus \{x_0\}\} \neq \varnothing$.
Or equivalently, $\exists\, y \in E$ such that $y \neq x_0$ and $y \in N_r(x_0)$, i.e. $d(x, y) < r$.

Note Theorem 2.20 relates to the definition of limit point.
See proof below.

(c) If $x \in E$ is not a limit point of E, then x is an *isolated point* of E.
Or, equivalently, $\exists\, r > 0$ such that $N_r(x_0) \cap \{E \setminus \{x_0\}\} = \varnothing$.

Example $E = (0, 1) \cup \{2\}$. 2 is an isolated point of E.

(d) $E \subset X$ is *closed* if every limit point of E belongs to E.
(e) $x \in E$ is an *interior point* of E if $\exists\, r > 0$ such that $N_r(x) \subset E$.
(f) $E \subset X$ is *open* if every point of E is an interior point of E.

Note E is open $\Leftrightarrow X \setminus E$ is closed. And E is closed $\Leftrightarrow X \setminus E$ is open.

Theorem 2.20 If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Sketch of proof (informal):
Let $r > 0$.
By the definition of limit point
$\exists\, x_1 \neq x_0$ where $x_1 \in N_r(x_0)$.
By the definition again
$\exists\, x_2 \neq x_0$ where $x_2 \in N_{x_0 - x_1}(x_0)$.
We continue this infinitely many times to find infinitely many points in $N_r(x_0)$.
See diagram.