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(a) If $\{G_\alpha\}$ are open then $\bigcup_\alpha G_\alpha$ is open too.

(b) If $\{F_\alpha\}$ are closed then $\bigcap_\alpha F_\alpha$ is closed too.

(c) If $G_1, G_2, ..., G_n$ are open then $\bigcap_{i=1}^n G_i$ is open.

(d) If $F_1, F_2, ..., F_n$ are closed then $\bigcup_{i=1}^n F_i$ is closed.

Homework Discussion

Correction to **#1.** Suppose $B \neq \varnothing$, $B \subset A$, and $f: A \to B$ is 1-1. Let $C_0 = A \setminus B$. Assume $C_0 \neq \varnothing$. Let $C_{n+1} = f(C_n)$. Let $C = \cup_{n\in\mathbb{N}} C_n$.

Then $f(C) = C \setminus C_0 \leftarrow$ This is the correction.

In **Exercise 6**, E' denotes the set of limit points of E. Is $(E')' = E'$?

If $E = (a, b)$, then $E' = [a, b]$. It appears $(E')' = E'$ here.

Consider $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Then $E' = \{0\}$.

In **Exercise 5**, $X = C[0, 1]$.

This is notation for $X = \{f: [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}$.

Define $d(f, g) = \max_{x\in[0,1]}|f(x) - g(x)|$.

In **Exercise 7**, (X, d_X) and (Y, d_Y) are metric spaces.

Define a new metric space, $(X \times Y, d)$ by

$d((x_1, y_1), (x_2, y_2)) = d_X((x_1, x_2) + d_Y((y_1, y_2)$.

In part **(b)** Let $U \subset X \times Y$.

Prove if U is open, then U_X is open where U_X is the projection of U onto the x axis.

If U_X is open, then is U open? If U is closed then is U_X closed?

Definitions **2.18** (X, d) is a metric space.

(a) For $r > 0$, $x \in X$, $N_r(x) = \{y \in X | \, d(x, y) < r\}$.
(b) x is a limit point of $E \subset X$ if $\forall\, r > 0$, $N_r(x) \cap \{E \setminus \{x\}\} \neq \varnothing$.
(c) $x \in E$ is an isolated point if it is not a limit point.
(d) E is closed if it contains all of its limit points.
(e) $x \in E$ is an interior point if $\exists\, r > 0$ such that $N_r(x) \subset E$.
(f) E is open if all of its points are interior points.

Example $E = (0, 1)$ is open.
$E \subset \mathbb{R}$ is open if it is a union of open intervals, that is
$E \subset \mathbb{R}$ is open if $E = \bigcup_{n=1}^{\infty} E_n$ where $E_n = (a_n, b_n)$.

Note Closed sets are more interesting because they cannot be described as simply as open sets, such as the Cantor Set.

(g) $E^c = X \setminus E$.
(h) E is perfect if every point of E is a limit point of E and E is closed.

Examples $E = [a, b]$ is perfect.
$[0, 1] \cap \mathbb{Q} \subset \mathbb{R}$ is not perfect as the irrationals are limit points of [0, 1].
If $X = \mathbb{Q}$, then $[0, 1] \cap \mathbb{Q}$ is perfect.

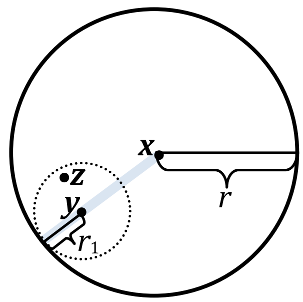
(i) E is bounded if $\exists\, x \in X$ and $\exists\, r > 0$ such that $E \subset N_r(x)$.
(j) E is dense in X if every $x \in X$ is a limit point of E or an element of E.
or, alternatively, E is dense in X if $X = \overline{E}$ where $\overline{E} = E \cup E'$.

Example $\mathbb{Q}$ is dense in $\mathbb{R}$.

Recall Theorem 1.20 ($\forall\, x, y \in \mathbb{R}$, $\exists\, p \in \mathbb{Q}$ such that $x < p < y$).
We can apply this theorem to construct a sequence $\{p_n\}$.
Let $x_0 \in \mathbb{R}$. Then
$\exists\, p_1$ such that $x_0 < p_1 < x_0 + 1$, and
$\exists\, p_2$ such that $x_0 < p_2 < p_0$,
$\vdots$
$\exists\, p_n$ such that $x_0 < p_n < p_{n-1}$.
Does $p_n \to x_0$? No. If $x_0 = 0$ and $p_n = 1/2 + 1/(n+2)$, then $p_n \to 1/2$.
To fix this, construct the sequence as follows:
$\exists\, p_1$ such that $x_0 < p_1 < x_0 + 1$
$\exists\, p_2$ such that $x_0 < p_2 < x_0 + 1/2$
$\exists\, p_3$ such that $x_0 < p_3 < x_0 + 1/3$
$\vdots$
$\exists\, p_n$ such that $x_0 < p_n < x_0 + 1/n$. In this case $p_n \to x_0$.

Theorem **2.19** $\forall\, x \in X, \forall\, r > 0, N_r(x)$ is open.

Proof:
Let $y \in N_r(x)$. Let $r_1 = r - d(x, y)$.
Let $z \in N_{r_1}(y)$.
Then $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r$.
$\therefore\ N_{r_1}(y) \subset N_r(x)$, hence y is an interior point of $N_r(x)$.



Theorem **2.20** If $x \in X$ is a limit point of E then $N_r(x)$ contains infintely many points of E for all $r > 0$. (See lecture notes 10/6/10.)

Corollary A finite set has no limit points.

Theorem **2.22** DeMorgan's Laws $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$.
See text for proof. This is a straight set theoretical proof.

Theorem **2.23** $E \subset X$ is open $\Leftrightarrow E^c$ is closed.

Proof:
$\Rightarrow$: Assume E is open. Let $x \in X$ be a limit point of E^c.
We will show $x \in E^c$, hence E^c is closed.
$\forall\, r > 0, N_r(x) \cap \{E^c \setminus \{x\}\} \neq \varnothing$ by defn limit point of E^c.
$\Rightarrow \forall\, r > 0, N_r(x) \cap E^c \neq \varnothing$
$\Rightarrow N_r(x) \not\subset E$
$\Rightarrow x \notin E$
$\Rightarrow x \in E^c$.

$\Leftarrow$: Assume E^c is closed. Let $x \in E$.
We will show x is an interior point of E, hence E is open.
$x \in E \Rightarrow x \notin E^c = E^c \cup E^{c\prime}$ as E^c is closed
$\Rightarrow x$ is not a limit point of E^c
$\Rightarrow \exists\, r > 0$ such that $N_r(x) \cap \{E^c \setminus \{x\}\} = \varnothing$
$\Rightarrow N_r(x) \cap E^c = \varnothing$
$\Rightarrow N_r(x) \subset E$.
$\Rightarrow$ x is an interior point of E.

Theorem **2.24**

(a) If $\{G_\alpha\}$ are open then $\bigcup_\alpha G_\alpha$ is open too.

(b) If $\{F_\alpha\}$ are closed then $\bigcap_\alpha F_\alpha$ is closed too.

(c) If $G_1, G_2, \ldots, G_n$ are open then $\bigcap_{i=1}^n G_i$ is open.

(d) If $F_1, F_2, \ldots, F_n$ are closed then $\bigcup_{i=1}^n F_i$ is closed.

Remark Properties of Open Sets

(a) $\varnothing$ and X are open.

(b) $\{G_\alpha\}$ open $\Rightarrow \bigcup_\alpha G_\alpha$ is open.

(c) $G_1, G_2, \ldots, G_n$ are open $\Rightarrow \bigcap_{i=1}^n G_i$ is open.

Example Let $X = \mathbb{R}$. Consider the topology on X, "Any subset is open."

With the metric $d(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$, we have all subsets are open.

Since $N_{1/2}(x) = \{x\}$, then every single element set is open.
Every subset is closed also as the complement of every open set, $\{x\}$, is open, hence $\{x\}$ is closed.