Content:

Theorem	2.30 E is open relative to $Y \Leftrightarrow \exists\, G \subset X$, open in X such that $E = Y \cap G$.
Definition	Open cover, compact
Theorem	2.33 Suppose $K \subset Y \subset X$. Then K is compact relative to $Y \Leftrightarrow K$ is compact relative to X.
Theorem	2.34 Compact subsets of metric spaces are closed.

Homework Discussion

C = Cantor Set. In addition to other properties, interior $C = \varnothing$.
We showed previously:
1. Construction of Cantor Set
2. Representation of Cantor Set elements in base 3.

$x = a_1a_2a_3...$ where $a_i \in \{0, 2\}$. If 1/3 = x, then x = 0.1 = 0.222...
But 0.021 is ok.

3. Cantor Set is uncountable.
4. Cantor Set is closed.
5. Cantor Set is perfect.

And now we have

6. interior of Cantor Set is $\varnothing$.

If it wasn't, then for some $x_0 \in C$, we would have $(x_0 - \varepsilon, x_0 + \varepsilon) \subset C$.
Let T = $\{0 \le \frac{k}{3^n} \le 1 \mid k, n \in \mathbb{N}\}$. Note that T is dense in [0, 1].
Thus, for some k, $\exists\, n \in \mathbb{N}$ such that $\frac{k}{3^n} \in (x_0 - \varepsilon, x_0 + \varepsilon) \subset C$.
If $\frac{k}{3^n} \notin C$, then we're done.
If not, then $\exists\, m > n$ such that $\frac{k}{3^n} + \frac{1}{3^m} \in (x_0 - \varepsilon, x_0 + \varepsilon)$. And $\frac{k}{3^n} + \frac{1}{3^m} \notin C$ as it contains a 1 as its last digit in its base 3 representation.
(Preceding proof by PlanetMath).

1(b) If we vary the Cantor Set construction by first removing the middle 3/5 of [0, 1], then we have 1/10 remaining on each side.
It turns out all the properties remain the same except for the ternary representation becomes a base 10 representation and the Hausdorf dimension is smaller.

2. Theorem 2.28 repeats the exercise from Week #1, #1. That is,
If $\sup A \notin A$, then $(\sup A - \varepsilon, \sup A)$ contains infinitely many points of A.

3. (b) Better to wait until we have covered sequences before doing this problem.
(c) Does $\overline{S}$ convex $\Rightarrow$ S convex? No.
Consider a ball with deleted center. The center is included in the closure, hence $\overline{S}$ is convex by part (a). Yet, S is not convex.

Note Convex Function

Suppose $S \subset \mathbb{R}^n$. The idicator function $i_S : \mathbb{R}^n \to \mathbb{R}$ is defined by

$i_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ if } x \notin S \end{cases}$. The indicator function is convex $\Leftrightarrow$ S is convex.

#4, 5, 6 have been chosen in a grouped fashion.

7. Prove an isolated set is at most countable.

Each isolated point has a neighborhood that contains only the point.

By the density property, we can find a point with rational coordinates that is contained in each isolated point's neighborhood.

8. Use the same idea as for #7.

9. Show $(0, 1) \times (0, 1) \sim (0, 1)$. You think about it.

Remark 2.29 (X, d) is a metric space. Y is a nonempty subset of X.

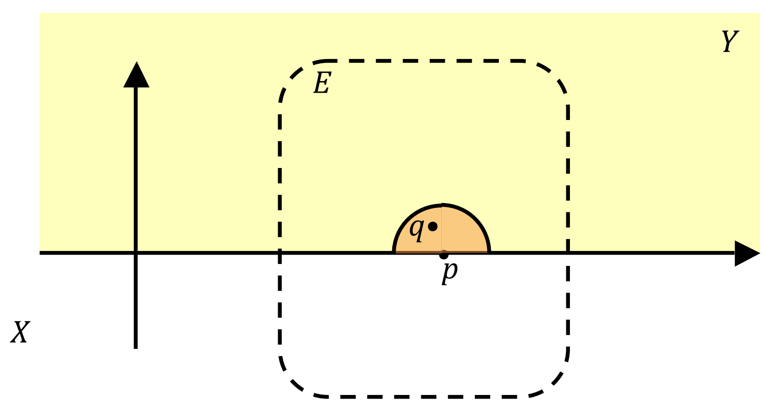
Then (Y, d) is also a metric space. If $E \subset Y$, then we can view E relative to X or E relative to Y.

We say E is open relative to Y if

$\forall\, p \in E, \exists\, r > 0$ such that $\forall\, q \in Y$ with $d(p, q) < r$, $q \in E$.

In other words, $N_r(p) \cap Y \subset E$.

Example Consider $X = \mathbb{R}^2$, $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$ This is the upper half-plane.



Theorem 2.30 E is open relative to $Y \Leftrightarrow \exists\, G \subset X$, open in X such that $E = Y \cap G$.

Read the proof in the text.

Definition 2.31 (X, d) is a metric space. An *open cover* for $E \subset X$ is a collection of open sets $\{G_\alpha\}$ such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.32 $E \subset X$ is *compact* if from every open cover we can extract a finite subcover.

Example Let $E = [0, 1]$. $\forall x \in E$, let $G_x = (x - 1/4, x + 1/4)$.
Then $\{G_x \mid 0 \le x \le 1\}$ is an open cover. We can extract a finite cover from this set. $G_0 \cup G_{1/5} \cup G_{2/5} \cup G_{3/5} \cup G_{4/5}$ covers $[0, 1]$.

Example Let $E = [0, 1)$. Then E is not compact.
Let $G_n = (-1, \frac{2^n - 1}{2^n})$. Then $\{G_n \mid n \in \mathbb{N}\}$ is an open cover of $[0, 1)$.
But there is no finite subcover of $\{G_n\}$, for
$G_{n_1} \cup G_{n_2} \cup \cdots \cup G_{n_k} = (-1, \frac{2^{n_k} - 1}{2^{n_k}}) \neq [0, 1)$.

Example Let $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ not compact.
E has infinitely many isolated points.
Let $G_n = \left(\frac{1}{n} - \frac{1}{n+2}, \frac{1}{n} + \frac{1}{n+2}\right)$.
Then $\{G_n\}$ is an open cover of E with no finite subcover.

Theorem 2.33 Suppose $K \subset Y \subset X$. Then K is compact relative to $Y \Leftrightarrow K$ is compact relative to X.

Proof:
$\Rightarrow$: Suppose K is compact relative to Y.
Consider an open cover of K in X.
We will show it contains a finite subcover of K.
Let G_α be open in X and $K \subset \bigcup_\alpha G_\alpha$.
So then $\{G_\alpha\}$ is an open cover of K in X.
Since $K \subset Y$, then $K \subset \bigcup_\alpha (G_\alpha \cap Y)$.
$G_\alpha \cap Y$ is open relative to Y. That is, $G_\alpha \cap Y$ is open in Y.
So then $\{G_\alpha \cap Y\}_\alpha$ is an open cover of K in Y.
So $\exists$ n such that $K \subset \bigcup_{i=1}^n (G_{\alpha_i} \cap Y) \Rightarrow K \subset \bigcup_{i=1}^n G_{\alpha_i}$ in X.

$\Leftarrow$: Suppose K is compact relative to X.
Consider an open cover of K in Y.
Let E_α be open in Y where $K \subset \bigcup_\alpha E_\alpha$.
By Theorem 2.30 above, for each α, $\exists$ G_α open relative to X such that $E_\alpha = G_\alpha \cap Y$. So then $K \subset \bigcup_\alpha (G_\alpha \cap Y)$.
Since K is compact relative to X, and $(G_\alpha \cap Y) \subset G_\alpha$, then $K \subset \bigcup_{i=1}^n G_{\alpha_i}$.
And $K \subset Y \Rightarrow K \subset \bigcup_{i=1}^n (G_{\alpha_i} \cap Y) = \bigcup_{i=1}^n E_{\alpha_i}$.
$\therefore$ K is compact relative to Y.

Theorem **2.34** Compact subsets of metric spaces are closed.

Proof:
Let (X, d) be a metric space. Let E be a nonempty compact subset of X.

Consider $X \setminus E$. We will prove $X \setminus E$ is open.

Let $x \in X \setminus E$. For $y \in E$, choose $r_y = d(x, y)/3$.

Then $\{G_y = N_{r_y}(y) \mid y \in E\}$ is an open cover of E.

By compactness, there is a finite subcover, $G_{y_1}, G_{y_2}, \ldots, G_{y_n}$.

Let $G = \bigcup_{i=1}^{n} G_{y_i}$. Let $r = \min(r_{y_i} \mid 1 \leq i \leq n\}$.

Then $N_r(x) \subset X \setminus E$. Thus, x is an interior point of $X \setminus E$.

So, by the fact that x was chosen arbitrarily, we get $X \setminus E$ is open.

Hence E is closed.