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Theorem 2.35 Closed subsets of compact sets are compact.

Corollary If F is closed and K is compact, then $F \cap K$ is compact.

Theorem 2.36 If $\{K_\alpha\}$ is a collection of compact sets such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty,then $\cap_\alpha K_\alpha \neq \varnothing$.

Corollary If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_{n+1} \subset K_n \ \forall \ n \in \mathbb{N}$, then $\cap_{1 \leq n \leq \infty} K_n \neq \varnothing$.

Theorem 2.37 If E is an infinite subset of a compact set K, then E has a limit point.

Homework Discussion Exercise 18, p. 44 You cannot have a perfect set that contains only rationals as perfect sets are uncountable.
Choose $[\sqrt{2}, \sqrt{2}+1]$. We cannot have rationals in this set, but if we delete, say, 2, then the irrationals that have limit 2 will have no limit in the set. So we can construct this set in a manner similar to constructing the Cantor Set. We can delete successive intervals.
First order the rationals of the set, $\{q_1, q_2, ..., q_{100}, q_{101}, ...\}$.
Then delete the interval that contains the first n rational numbers, leaving irrational endpoints.
Any Cantor-type construction will produce a perfect set.

Review (X, d) is a metric space.
$K \subset X$ is compact $\Leftrightarrow$ every open cover has a finite subcover.
Compact sets are closed in metric spaces.

Theorem **2.35** Closed subsets of compact sets are compact.

Proof:
Consider $K \subset X$ compact and $\varnothing \neq K$.
Consider $F \subset K$ closed. Let $\{G_\alpha\}$ be an open cover of F.
Note that F is closed $\Rightarrow X \setminus F$ is open.
$\{G_\alpha\} \cup \{X \setminus F\}$ is an open cover of K.
K is compact so we can select $X \setminus F, G_1, G_2, ..., G_n$.
$F \subset K \subset [(X \setminus F) \cup G_1 \cup G_2 \cup \cdots \cup G_n]$.
Thus, $F \subset G_1 \cup G_2 \cup \cdots \cup G_n$ (as $F \cap (X \setminus F) = \varnothing$).
And this is a finite subcover of $\{G_\alpha\}$.

Corollary If F is closed and K is compact, then $F \cap K$ is compact.

Proof:
$F \cap K$ is closed as K is compact $\Rightarrow K$ is closed. So $F \cap K$ is a finite intersection of closed sets. And $(F \cap K) \subset K$, so then by Thm 2.35 $F \cap K$ is compact.

Theorem **2.36** Important

If $\{K_\alpha\}$ is a collection of compact sets such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty,then $\cap_\alpha K_\alpha \neq \varnothing$.

Proof:

Suppose (X, d) is compact. If not, then replace X by $X \cap K_{\alpha_0}$ for a fixed α_0.
Say $X = [0, 1 + 1/n]_{n\in\mathbb{N}}$. Then X is not compact, but each K_n is.
So then we can just replace X with $X \cap K_1$. And replace each K_α with $K_\alpha \cap K_1$.
Suppose $\cap_\alpha K_\alpha = \varnothing$. Let $G_\alpha = X \setminus K_\alpha$ fo each α.
Then $G_\alpha = X \setminus K_\alpha$ form an open cover of X by DeMorgan's Laws.
$\cup G_\alpha = \cup(X \setminus K_\alpha) = X \setminus \cap K_\alpha = X \setminus \varnothing = X$.
Use the fact that X is compact and extract a finite subcover
$G_1 \cup G_2 \cup \cdots \cup G_n = X \setminus \cap_{1\le i\le k} K_i = X$.
So $\cap_{1\le i\le n} K_i = \varnothing$, contrary to our hypothesis that every finite subcollection has a nonempty intersection.

Alternative proof:

Fix K_1 and suppose that K_1 does not interesect with any K_α (i.e. $\cap_\alpha K_\alpha = \varnothing$).
Let $\{K_\beta\} = \{K_\alpha\} \setminus \{K_1\}$. Then we have

$K_1 \cap (\cap_\beta K_\beta) = \varnothing \Rightarrow K_1 \subset (\cap_\beta K_\beta)^c$,
$\Rightarrow K_1 \subset \cup_\beta K_\beta{}^c$, by DeMorgan's Laws,
$\Rightarrow K_1 \subset \cup_{1\le i\le n} K_i{}^c$, as K_1 is compact,
$\Rightarrow K_1 \subset (\cap_{1\le i\le n} K_i)^c$, by DeMorgan's Laws,
$\Rightarrow K_1 \cap (\cap_{1\le i\le n} K_i) = \varnothing$, contrary to our hyp.

Corollary If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_{n+1} \subset K_n \ \forall\ n \in \mathbb{N}$, then $\cap_{1\le n\le\infty} K_n \neq \varnothing$.

Example If $K_n = [-1/n, 1/n]$, then $\cap_{n\in\mathbb{N}} K_n = \{0\}$.
If $K_n = [-1/n, 1 + 1/n]$, then $\cap_{n\in\mathbb{N}} = [0, 1]$.
Thus, the intersection can be unique or can have many values.

Theorem **2.37** Important

If E is an infinite subset of a compact set K, then E has a limit point.

Proof:

Suppose E does not have any limit points.
Then $\forall\ x \in K, \exists\ r_x$ such that $N_{r_x}(x)^* \cap E = \varnothing$.
$\{N_{r_x}(x) \mid x \in K\}$ is an open cover of K.
This does not have any finite subcover, contrary to compactness of K.

Other proof:

In $\mathbb{R}^n$, we can cut our set into 4 parts. One of them contains infinitely many points. The fourth that contains infinitely many points can be cut into 4 parts. And so on.