Content:

Theorem 2.41 If $E \subset \mathbb{R}^n$, then the following are equivalent:
(a) E is closed and bounded, (b) E is compact,
(c) Every infinite subset of E has a limit point in E.

Homework Discussion

Regarding **Week #7, Exercise 8.**
The topology on $\mathbb{R}$ is 2nd countable.
If $S \subset \mathbb{R}$, then every open cover has a countable subcover.

Proof:
$\forall x \in S, \exists r_x > 0$ such that $N_{r_x}(x) \cap S$ is countable.
$\{N_{r_x}(x) | x \in S\}$ is an open cover of S.
So we can select $x_1, x_2, ..., x_{n_1}$ such that $S \subset \bigcup_{i=1}^{\infty}\left[N_{r_{x_i}}(x_i) \cap S\right]$.
Thus, S is countable.

Alternate proof:
Suppose $S \subset \mathbb{R}$ is uncountable.
We can cover S with $\{[n, n+1] | n \in \mathbb{N}\}$.
So $\exists\ [a, b]$ such that $S \cap [a, b]$ is uncountable.
Let $T = S \cap [a, b]$. Then $T \subset [a, b]$.
Since $[a, b]$ is closed, then $\overline{T} \subset [a, b]$.
By Theorem 2.41, $[a, b]$ is compact, so then by Theorem 2.35.
$\overline{T}$ is compact.
Question: Is $\overline{T} \subset \bigcup_{x \in T} N_{r_x}(x)$? See if you can finish this proof.

Week #8, Exercise 1 (a)
Let E be an infinite subset of a compact set K.
Assume E has no limit point in K.
Then E contains all of its limit points (of which it has none),
hence E is closed.
And by Thm 2.35 (*Closed subsets of compact sets are compact.*), E is compact.
Then $\forall x \in E, \exists r_x$ such that $N_{r_x}(x) \cap E = \{x\}$.
$\{N_{r_x}(x) | x \in E\}$ is an open cover of E.
Since E is compact, then $\exists \{N_{r_{x_i}}(x) \cap E \mid i = 1, 2, ..., n\}$ that covers E.
$\therefore$ E is finite, contrary to our hypothesis that E is infinite.

Week #8, Exercise 1 (b)

Let E be an infinite subset of a compact set K.

$\{N_1(x) | x \in K\}$ is an open cover of K. Since K is compact,

$\exists \{N_1(x_i)\}_{1 \le i \le n} \subset \{N_1(x) | x \in K\}$, a finite subcover of K.

Since E has infinitely many points, then $\exists\, i_1$ such that $N_1(x_{i_1})$ has infinitely many points of E.

Let $K_1 = \overline{N_1(x_{i_1})} \cap K$.

By corollary to Thm 2.35 (*If F is closed and K is compact, then F ∩ K is compact.*), K_1 is compact.

$\{N_{1/2}(x) | x \in K_1\}$ is an open cover of K_1. Since K_1 is compact,

$\exists \{N_{1/2}(x_i)\}_{1 \le i \le n'} \subset \{N_{1/2}(x) | x \in K_1\}$, a finite subcover of K_1.

Since K_1 has infinitely many points of E, then $\exists\, i_2$ such that $N_{1/2}(x_{i_2}) \cap K_1$ has infinitely many points of E.

Let $K_2 = \overline{N_{1/2}(x_{i_2})} \cap K_1$.

Continuing the construction in this manner, we have

$K_n = \overline{N_{1/n}(x_{i_n})} \cap K_{n-1}$.

Notice that $K_n \subset K_{n-1} \subset \cdots \subset K_1 \subset K$ and $\forall\, n$, K_n contains infinitely many points of E. Thus, we have $\{K_\alpha\}$, a collection of compact subsets of K such that every finite subcollection of $\{K_\alpha\}$ is nonempty.

By Theorem 2.36, $\cap K_\alpha \neq \varnothing$.

Let $w \in \cap K_\alpha$. Then $\forall\, \varepsilon > 0$, $\exists\, n$ such that $1/n < \varepsilon$ and $w \in \overline{N_{1/n+1}(x_{i_{n+1}})} \cap K_n$.

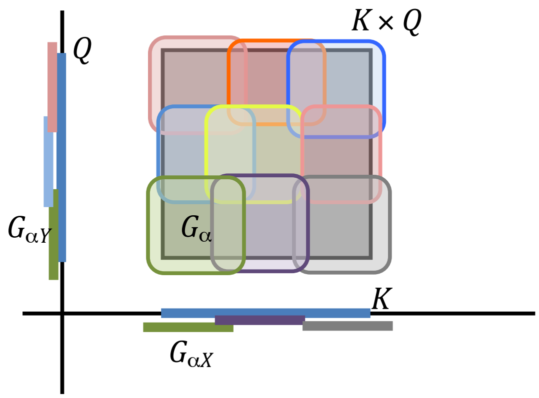
Since $\overline{N_{1/n+1}(x_{i_{n+1}})} \cap K_n$ contains infinitely many points of E, then $\exists\; z \in \overline{N_{1/n+1}(x_{i_{n+1}})} \cap K_n$ such that $z \in E$ and $z \neq w$.

And $0 \le d(z, w) < 1/(n+1) < 1/n < \varepsilon$. $\therefore\; d(z, w) = 0$, and so $z = w$.

Thus $z \in N_\varepsilon(w)$, hence $N_\varepsilon(w)^* \cap E \neq \varnothing$.

$\therefore\; w$ is a limit point of E in K, as desired.

Exercise 2 (a)



Take an open cover $\{G_\alpha\}$ of $K \times Q$, then project each G_α onto X and onto Y. Then $K \subset \cup \{G_{\alpha X}\}$ and $Q \subset \cup\{G_{\alpha Y}\}$. Since K and Q are compact, then we can extract a finite subcover of $\{G_{\alpha X}\}$ and $\{G_{\alpha Y}\}$. The cross product of these subcovers cover $K \times Q$.

Exercise 2(b)

We need to show that $[a, b] \subset \mathbb{R}$ is compact.

Suppose it is not compact.

Then $\exists\, G_\alpha$ which does not have any finite subcover.

Let $a_0 = a$, $b_0 = b$. If $[a, (a+b)/2]$ does not have a finite subcover of G_α, then let $a_1 = a$, $b_1 = (a + b)/2$. Otherwise let $a_1 = (a + b)/2$, $b_1 = b$. Continue subdividing in this way to get $[a_n, b_n]$ does not have a finite subcover, $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$, and $b_n - a_n = (b - a) \bullet (1/2^n)$.

Notice that $a_0 \le a_1 \le \cdots a_n \le b_n \le \cdots \le b_1 \le b_0$.

If $A = \{a_n\}$, and $B = \{b_n\}$, then A is bounded above and B is bounded below.

So $\alpha = \sup A \le \inf B = \beta$ and $\alpha, \beta \in [a_n, b_n]$, $\forall\, n \in \mathbb{N}$.

Since each $[a_n, b_n]$ is closed and bounded, then each is compact.

Since for each n, $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$

Since the diameter of $[a_n, b_n] \to 0$ as $n \to \infty$, then $\alpha = \beta = x^*$.

Thus, $\exists\, \gamma^*$ such that $x^* \in G_{\gamma^*}$.

So $\exists\, n \in \mathbb{N}$ such that $x^* \in [a_n, b_n] \subset G_{\gamma^*}$.

This contradicts that for each n, $[a_n, b_n]$ was chosen so that $[a_n, b_n]$ does not have any finite subcover of G_α.

Note From this, Theorem 2.38 is proved.

For $\{I_n\}$ closed and bounded intervals where $I_n = [a_n, b_n]$ compact, and $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$ $\forall\, n \in \mathbb{N}$, $n \ge 2$, we have $\bigcap_{n \in N} [a_n, b_n] \ne \varnothing$.

Theorem 2.40 (cont.) Every k-cell is compact as finite cartesian products of compact intervals are compact.

Theorem **2.41**

If $E \subset \mathbb{R}^n$, then the following are equivalent:

(a) E is closed and bounded,

(b) E is compact,

(c) Every infinite subset of E has a limit point in E.

Proof:

(a) $\Rightarrow$ (b): $E \subset \mathbb{R}^n$ is bounded $\Rightarrow E \subset [a_1, b_1] \times \cdots \times [a_k, b_k]$, $a_i, b_i \in \mathbb{R}$.

E closed, $\Rightarrow E$ is compact, by Theorem 2.40 (*Every k-cell is compact.*)

(b) $\Rightarrow$ (c) We have this by Theorem 2.37 (*Every infinite set E of a compact set K has a limit point in K.*) Since we have by Theorem 2.34 (*Compact subsets of metric spaces are closed.*) that E is closed, then E contains its limit points, hence every infinite subset of E has a limit point in E.

(c) $\Rightarrow$ (a) We want to show $E' \subset E$.

Let $x \in E'$. $\forall\, \varepsilon > 0$, $N_\varepsilon(x) \cap E$ is infinite. Select $x_n \in N_{1/n}(x) \cap E$.

Then $\{x_n\}$ has x as its limit point. Then, by our hypothesis, $x \in E$.

To show E is bounded, suppose it is not.

Then consider $x_n \in E \cap N_n(0) \neq \varnothing$.

Then $\{x_n \mid n \in \mathbb{N}\} \subset E$ but doesn't have any limit points, contrary to our hypothesis.

Thus, E is closed and bounded.

Note Compact $\Rightarrow$ Closed and bounded always.

Closed and bounded $\Rightarrow$ compact only in $\mathbb{R}^n$.