Content:

Theorem	2.43 Let P be a perfect set in $\mathbb{R}^k$. Then P is uncountable.
Definition	2.45 Separated sets, connected sets
Theorem	2.47 $E \subset \mathbb{R}$ is connected $\Leftrightarrow \forall\, x, y \in E$, where $x < y$ and $\forall\, z \in \mathbb{R}$ such that $x < z < y$, if follows that $z \in E$.

Theorem **2.43** Let P be a perfect set in $\mathbb{R}^k$. Then P is uncountable.

Proof:
Suppose $P = \{x_1, x_2, x_3, \ldots\}$ (i.e. P is countable).
Since P is perfect, then every point of P is a limit point of P.
Let $r_1 > 0$. Start with $V_1 = N_{r_1}(x_1)$. Construct V_2 as follows:
(i) $\overline{V}_2 \subset V_1$; (ii) $x_1 \notin \overline{V}_2$; (iii) $P \cap V_2 \neq \varnothing$.
In general, construct V_{n+1} so that
(i) $\overline{V}_{n+1} \subset V_n$; (ii) $x_n \notin \overline{V}_{n+1}$; (iii) $P \cap V_{n+1} \neq \varnothing$.
Define $K_n = P \cap \overline{V}_n$. Then $K_{n+1} \subset K_n$.
Since each $\overline{V}_n$ is closed and bounded, then each is compact.
Thus $\cap_{n \in \mathbb{N}} K_n \neq \varnothing$, contrary to $x_n \notin \overline{V}_{n+1}$.

Note P is not necessarily bounded. $[0, +\infty)$ is perfect, but not bounded.

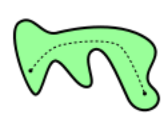
Definition 2.45 In a metric space X, 2 subsets A and B are separated if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

Examples $A = [0, 1)$ and $B = (1, 2)$ are separated as $[0, 1] \cap (1, 2) = [0, 1) \cap [1, 2] = \varnothing$.
$A = [0, 1]$ and $B = (1, 2)$ are not separated as $[0, 1] \cap [1, 2] \neq \varnothing$.

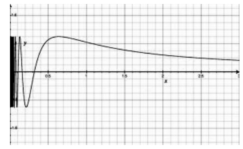
Definition $E \subset X$ is called connected if it cannot be written as $E = A \cup B$ where A, B are separated and $A \neq \varnothing$ and $B \neq \varnothing$.

Properties $E \subset X$ is connected $\Leftrightarrow$ it cannot be written as $E = A \cup B$ where $A, B \neq \varnothing$ $A \cap B = \varnothing$ and both A and B are open.
Proof:
$\Rightarrow$: Suppose $E = A \cup B$, if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.
Then $A \cap B \subset A \cap \overline{B} = \varnothing$, and we have
$A = E \cap (X \setminus \overline{B})$ and $B = E \cap (X \setminus \overline{A})$, both open sets.
$\Leftarrow$: Suppose $E = (A \cup B)$, $A, B \neq \varnothing$, $A \cap B = \varnothing$, and A and B are open.
To show $A \cap \overline{B} = \varnothing$, let $x \in A$ and note that $\exists$ r > 0 such that $N_r(x) \subset A$.
Since $B \subset \overline{B} \subset X \setminus A$, then $x \notin \overline{B}$. $\therefore$ $A \cap \overline{B} = \varnothing$.
Similarly $\overline{A} \cap B = \varnothing$.

Note E is pathwise connected if $\forall\, x, y \in E$, $x \neq y$, $\exists$ a curve inside E connecting x and y. (i.e there is a continuous function $\gamma: \mathbb{R} \to E$ st $\gamma(0) = x$, $\gamma(1) = y$.

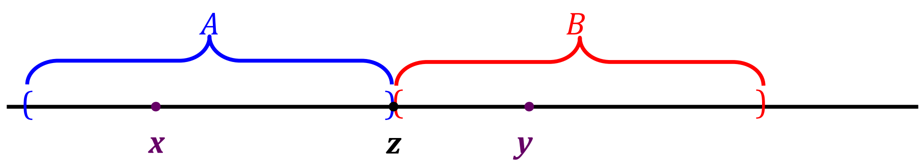


Example In $\mathbb{R}^2$ let $E = \{(x, \sin(1/x)) | \; 0 \le x \le \pi/2\} \cup \{[0, y] | \; -1 \le y \le 1\}$
Then E is connected but not pathwise connected.



Theorem **2.47** $E \subset \mathbb{R}$ is connected $\Leftrightarrow \forall \; x, y \in E$, where $x < y$ and $\forall \; z \in \mathbb{R}$ such that $x < z < y$, if follows that $z \in E$.

Proof:
$\Rightarrow$: Suppose $x, y \in E$ where $x < y$ and $z \in \mathbb{R}$ such that $x < z < y$, but $z \notin E$. Define $A = E \cap (-\infty, z)$ and $B = E \cap (z, \infty)$.
Then $A \cap \overline{B} = \varnothing = \overline{A} \cap B$, $A \cup B = E$, $A \neq \varnothing$ (as $x \in A$) and $B \neq \varnothing$ (as $y \in B$). So E is not connected.



$\Leftarrow$: Suppose that $E = A \cup B$, if $A \cap \overline{B} = \varnothing$, $\overline{A} \cap B = \varnothing$, $A \neq \varnothing$, and $B \neq \varnothing$.
Choose $x \in A$ and $y \in B$ (assuming wlog that $x < y$).
We will show $\exists \; z$ such that $z \notin E$.
Define $z = \sup(A \cap [x, y])$. We know z exists as A is bounded above by B and $[x, y]$ is bounded above by y.
$z = \sup(A \cap [x, y]) \Rightarrow z \in \overline{A \cap [x, y]} \subset \overline{A} \cap [x, y]$. So $z \in \overline{A}$, hence $z \notin B$.
Thus $z < y$. Also, $z = \sup(A \cap [x, y])$ and $x \in A \Rightarrow x \le z$.
To show $x < z$, notice that $z \in \overline{A} \Rightarrow z \in A$ or $z \notin A$.
If $z \notin A$, then $x \neq z$, hence $x < z$. Thus $z \notin E$.
If $z \in A$, then $z \notin \overline{B}$, hence $z \in \overline{B}^c$, an open set.
Thus $\exists \; r > 0$ such that $N_r(z) \subset \overline{B}^c$. By the density property $\exists \; z_1 \in \mathbb{R}$ such that $z < z_1 < z + r < y$.
And $z = \sup(A \cap [x, y]) \Rightarrow z_1 \notin A \cap [x, y]$; $z_1 \in [x, y] \Rightarrow z_1 \notin A$, hence $z \neq z_1$.
Thus $x < z_1 < y$ and $z_1 \notin E$.
$\therefore$ If E is connected, then $\forall \; x, y \in E$, $x < z < y \Rightarrow z \in E$.