Content:

Theorem 3.4 (a) In $X = \mathbb{R}^k$, $\mathbf{x}_n \to \mathbf{x} \Leftrightarrow x_{i\,n} \to x_i$ $(1 \le i \le k)$ where $\mathbf{x} = (x_1, x_2, ..., x_k)$.
(b) If $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, and $\beta_n \to \beta$ ($\{\beta_n\} \subset \mathbb{R}$), then
(i) $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$; (ii) $\mathbf{x}_n \bullet \mathbf{y}_n = \mathbf{x} \bullet \mathbf{y}$; and (iii) $\beta_n\mathbf{x}_n \to \beta\mathbf{x}$.

Definition 3.5 Subsequence

Theorem 3.6 (a) Let (X, d) be a compact metric space. If $\{p_n\} \subset X$, then there is at least one convergent subsequence of $\{p_n\}$.
(b) Every bounded sequence in $\mathbb{R}^k$ has at least one convergent subsequence of $\{p_n\}$.

Theorem 3.7 The set of subsequential limit points is closed.

Definition 3.8 Cauchy Sequence

Lemma 1 If $\{p_n\}$ is a Cauchy sequence, then it is bounded.

Lemma 2 Let $\{p_n\}$ be a Cauchy sequence. If there is a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to p \in X$ then $p_n \to p$.

Theorem 3.3 See Lecture Notes (11/8/10) for completion of proof.

Theorem **3.4** **(a)** In $X = \mathbb{R}^k$, $\mathbf{x}_n \to \mathbf{x} \Leftrightarrow x_{i\,n} \to x_i$ $(1 \le i \le k)$ where $\mathbf{x} = (x_1, x_2, ..., x_k)$.
(b) If $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, and $\beta_n \to \beta$ ($\{\beta_n\} \subset \mathbb{R}$), then
(i) $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$; **(ii)** $\mathbf{x}_n \bullet \mathbf{y}_n = \mathbf{x} \bullet \mathbf{y}$; and **(iii)** $\beta_n\mathbf{x}_n \to \beta\mathbf{x}$.

Proof:

(a) $\Rightarrow$: Assume $\mathbf{x}_n \to \mathbf{x}$. Let $\varepsilon > 0$.

$$d(\mathbf{x}_n, \mathbf{x}) = \left(\sum_{i=1}^{k}\left(x_{i\,n} - x_i\right)^2\right)^{1/2} \ge \sqrt{\left(x_{i\,n} - x_i\right)^2} = |x_{i\,n} - x_i| \ (1 \le i \le k).$$

$\therefore \exists N_\varepsilon \in \mathbb{N}$ such that $\forall\, n > N_\varepsilon$, $|x_{i\,n} - x_i| < d(\mathbf{x}_n, \mathbf{x}) < \varepsilon$,
hence $x_{i\,n} \to x_i$ $(1 \le i \le k)$.

$\Leftarrow$: Assume $x_{i\,n} \to x_i$ $(1 \le i \le k)$. Then $\forall\, \varepsilon > 0$, $\exists\, N_{i\,\varepsilon} \in \mathbb{N}$ such that

$$\forall\, n > N_{i\,\varepsilon},\ |x_{i\,n} - x_i| < \frac{\varepsilon}{\sqrt{k}}.\ \ \therefore\ \forall\, n > N_{i\,\varepsilon},\ d(\mathbf{x}_n, \mathbf{x}) < \left(\sum_{i=1}^{k}\left(\frac{\varepsilon}{\sqrt{k}}\right)^2\right)^{1/2} = \varepsilon.$$

(b) Left as an exercise.

(i) To show $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$, we note that by part (a)
$\mathbf{x}_n \to \mathbf{x} \Rightarrow x_{i\,n} \to x_i$ $(1 \le i \le k)$ and $\mathbf{y}_n \to \mathbf{y} \Rightarrow y_{i\,n} \to y_i$ $(1 \le i \le k)$.
Let $\mathbf{z}_n = \mathbf{x}_n + \mathbf{y}_n$ and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then by Theorem 3.3 (a),
for each i, $z_{i\,n} = x_{i\,n} + y_{i\,n} \to x_i + y_i = z_i$.
Thus by part (a) again, $\mathbf{z}_n \to \mathbf{z}$, hence $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$.

(ii) To show $\mathbf{x}_n \bullet \mathbf{y}_n \to \mathbf{x} \bullet \mathbf{y}$, we note that by part (a)
$\mathbf{x}_n \to \mathbf{x} \Rightarrow x_{i\,n} \to x_i$ $(1 \le i \le k)$ and $\mathbf{y}_n \to \mathbf{y} \Rightarrow y_{i\,n} \to y_i$ $(1 \le i \le k)$.
Let $\mathbf{z}_n = \mathbf{x}_n \bullet \mathbf{y}_n$ and let $\mathbf{z} = \mathbf{x} \bullet \mathbf{y}$, then by Theorem 3.3 (c),
for each i, $z_{i\,n} = x_{i\,n} \bullet y_{i\,n} \to x_i \bullet y_i = z_i$.
And by induction on Theorem 3.3 (a) $\mathbf{z}_n = \Sigma_{i=1}^{k} z_{i\,n} \to \Sigma_{i=1}^{k} z_i = \mathbf{z}$.
So then $\mathbf{x}_n \bullet \mathbf{y}_n = \mathbf{z}_n = \mathbf{z} = \mathbf{x} \bullet \mathbf{y}$.

(iii) To show $\beta_n\mathbf{x}_n \to \beta\mathbf{x}$, we note that by part (a)
$\mathbf{x}_n \to \mathbf{x} \Rightarrow x_{i\,n} \to x_i$ $(1 \le i \le k)$.
Let $\mathbf{z}_n = \beta_n\mathbf{x}_n$ and let $\mathbf{z} = \beta\mathbf{x}$. Then by Theorem 3.3 (c),
for each i, $\beta_n z_{i\,n} = \beta_n x_{i\,n} \to \beta x_i = \beta z_i$.
Thus by part (a) again, $\beta_n\mathbf{x}_n = \mathbf{z}_n \to \mathbf{z} = \beta\mathbf{x}$.

Subsequences

Definition **3.5** Consider a sequence $\{p_n\}_{n\in\mathbb{N}}$. Consider a sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ where $n_1 < n_2 < \cdots$. Then $\{p_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{p_n\}_{n\in\mathbb{N}}$.
If $p_{n_k} \to p$ then p is called a subsequential limit point of $\{p_n\}$.

Example Let $p_n = (-1)^n$. Let $n_k = 2k$. Then $p_{n_k} = p_{2k} = 1$.

Theorem **3.6 (a)** Let (X, d) be a compact metric space. If $\{p_n\} \subset X$, then there is at least one convergent subsequence of $\{p_n\}$.
(b) Every bounded sequence in $\mathbb{R}^k$ has at least one convergent subsequence of $\{p_n\}$.

Proof:
(a) *Case 1*: We have just finitely many different terms in $\{p_n\}$,
that is $\{p_n\} = \{p_1, p_2, \ldots, p_n\}$.
Then there is $\{p_{n_k}\}$ such that $p_{n_k} = p\ \forall\ k \in \mathbb{N}$.
For example, for $n_k = n+k$ we could define $p_{n_1} = p_{n_2} = \cdots = p_1$.
Case 2: We have infinitely many different terms in $\{p_n\}$.
Then by Theorem 2.37 (*If E is an infinite subset of a compact set K, then E has a limit point.*) the set $\{p_1, p_2, \ldots\}$ has a limit point. And by Theorem 2.20 (*If p is a limit point of a set E, then every neighborhod of p contains infinitely many points of E.*)
we can construct p_{n_k} where for every $\varepsilon = 1/k$, $\exists\ p_{n_k} \in \{p_n\}$ such that $d(p_{n_k}, p) < 1/k$. This shows $p_{n_k} \to p$.
(b) Every bounded sequence can be included in a ball of $\mathbb{R}^k$.
The closure of this ball in $\mathbb{R}^k$ is compact by Theorem 2.41 (*If $E \subset \mathbb{R}^n$, then the following are equivalent: (a) E is closed and bounded, (b) E is compact, (c) Every infinite subset of E has a limit point in E.*)
Thus, by (a) this sequence has a convergent subsequence.

Theorem **3.7** The set of subsequential limit points is closed.

Proof:
Let $\{p_n\} \subset X$. Let E^* be the set of subsequential limit points.
Consider $p \in E^{*\prime}$.
Then since p is a limit point of E^*, $\exists\ \{q_m\} \subset E^*$ such that $q_m \to p$.
Let $\varepsilon > 0$. $\exists\ m \in \mathbb{N}$ such that $d(p, q_m) < \varepsilon/2$.
Since $q_m \in E^*$, then $\exists\ \{p_{n_k}\} \subset \{p_n\}$ such that $p_{n_k} \to q_m$, hence
$\exists\ n_i$ such that $d(p_{n_i}, q_m) < \varepsilon/2$.
Then $d(p_{n_i}, p) \le d(p_{n_i}, q_m) + d(q_m, p) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
Thus p is a limit point of the subsequence $\{p_{n_i}\}$, hence $p \in E^*$.

Cauchy Sequences

Definition **3.8** In a metric space (X, d), a sequence $\{p_n\}$ is a Cauchy Sequence if $\forall\ \varepsilon > 0, \exists\ N_\varepsilon \in \mathbb{N}$ such that $d(p_n, p_m) < \varepsilon, \forall\ n, m \geq N_\varepsilon$.

Remark(1) If $\{p_n\}$ is a convergent sequence, then it is a Cauchy sequence.

Proof:
Suppose $p_n \to p$. Let $\varepsilon > 0$. $\exists\ N_\varepsilon \in \mathbb{N}$ such that $d(p_n, p) < \varepsilon/2, \forall\ n > N_\varepsilon$.
If $n, m > \varepsilon$, then $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Remark(2) In general, a Cauchy sequence is not convergent. However, in $\mathbb{R}^k$ the answer is yes. Prove in $\mathbb{R}^k$ if $\{p_n\}$ is Cauchy, then it is convergent.

Lemma 1 If $\{p_n\}$ is a Cauchy sequence, then it is bounded.

Proof:
Choose $\varepsilon = 1$. Then $\exists\ N_1 \in \mathbb{N}$ such that $d(p_n, p_m) < 1\ \forall\ n, m > N_1$.
Consider $R = \max\{1, d(p_1, p_n)_{1 \leq n \leq N_1 + 1}\}$.
Then $\{p_n\} \subset B_{R+1}(p_1)$.

Lemma 2 Let $\{p_n\}$ be a Cauchy sequence. If there is a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to p \in X$ then $p_n \to p$.

Proof:
Let $\varepsilon > 0$. Then $\exists\ N_\varepsilon \in \mathbb{N}$ such that $d(p_n, p_m) < \varepsilon/2, \forall\ n, m > N_\varepsilon$.
And $\exists\ K_\varepsilon \in \mathbb{N}$ such that $d(p, p_{n_k}) < \varepsilon/2, \forall\ k > K_\varepsilon$.
Let $N = \max\{N_\varepsilon, n_{K_\varepsilon}\}$.
Then $\forall\ n, n_k > N$, $d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
Thus $p_n \to p$.