Content:

Theorem 3.11 (a) Every convergent sequence is a Cauchy sequence.
(b) If (X, d) is a compact metric space and if $\{p_n\}$ is a Cauchy sequence, then $\{p_n\}$ converges in X.
(c) In $\mathbb{R}^k$, every Cauchy sequence is convergent.

Definition Complete metric space

Definition Monotone increasing/decreasing

Theorem(3.14) (a) If $\{s_n\} \subset \mathbb{R}$ is increasing and bounded from above, it is convergent.
(b) If $\{s_n\} \subset \mathbb{R}$ is decreasing and bounded from below, it is convergent.

Review Definition 3.8 In a metric space (X, d), a sequence $\{p_n\}$ is a Cauchy Sequence if $\forall\ \varepsilon > 0, \exists\ N_\varepsilon \in \mathbb{N}$ such that $d(p_n, p_m) < \varepsilon,\ \forall\ n, m \geq N_\varepsilon$.

Lemma 1 If $\{p_n\}$ is a Cauchy sequence, then it is bounded.
Lemma 2 Let $\{p_n\}$ be a Cauchy sequence. If there is a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to p \in X$ then $p_n \to p$.

Theorem **3.11 (a)** Every convergent sequence is a Cauchy sequence.
(b) If (X, d) is a compact metric space and if $\{p_n\}$ is a Cauchy sequence, then $\{p_n\}$ converges in X.
(c) In $\mathbb{R}^k$, every Cauchy sequence is convergent.

Proof:
(a) See Remark (1) Lecture Notes 11/10/10.
(b) Assume (X, d) is compact and $\{p_n\}$ is a Cauchy sequence.
Case 1: Just finitely many different terms exist in $\{p_n\}$.
Example: $X = [-1, 1]$, $p_n = (-1)^n$. This won't work.
Suppose $\exists\ x \neq y$ such that $p_{n_k} = x\ \forall\ k \in \mathbb{N}$ and $p_{n_i} = y\ \forall\ i \in \mathbb{N}$.
The problem here is that $x \neq y \Rightarrow d(x, y) > 0$, so for r small enough we can find $p_{n_k} \in B_r(x)$ such that $p_{n_k} \notin B_r(y)$.
The case of finitely many different terms works just if $p_n = p\ \forall\ n > N$ for some $N \in \mathbb{N}$. Then $p_n \to p$.
Case 2: There are infinitely many different terms in $\{p_n\}$.
Then by Theorem 2.37 (*If E is an infinite subset of a compact set K, then E has a limit point.*) there is a limit point, p, of these terms.
Construct a subsequence $p_{n_k} \to p$. Then use Lemma 2 (*Let $\{p_n\}$ be a Cauchy sequence. If there is a subsequence $\{p_{n_k}\}$ such that $p_{n_k} \to p \in X$ then $p_n \to p$.*) to show $p_n \to p$.
(c) Let $\{p_n\} \subset \mathbb{R}^k$ be a Cauchy sequence.
Lemma 1(*If $\{p_n\}$ is a Cauchy sequence, then it is bounded.*) gives us that $\{p_n\}$ is bounded, so $\exists\ R > 0$ such that $p_n \in \overline{B_R(0)}\ \forall\ n \in \mathbb{N}$.
Since is $\overline{B_R(0)}$ closed and bounded, then $\overline{B_R(0)}$ is compact.
So then by part (b) $\{p_n\}$ is convergent.

Remark In $\mathbb{R}^k$ the notion of Cauchy sequence is equivalent to the notion of convergent sequence.

Definition 3.12 A metric space (X, d) in which every Cauchy sequence is convergent is called a *complete* metric space.

Examples $X = \mathbb{R}^k$ is complete.
$X = [a, b]$ is complete.
$X = (a, b]$ is not complete as $x_n = a + 1/n$.
$X = \mathbb{Q}$ is not complete $x_n \to \sqrt{2} \notin \mathbb{Q}$.
$\mathscr{C}[0, 1]$ where $d(f, g) = \max_{0\leq x\leq 1}|f(x) - g(x)|$ is complete.

Proposition If (X, d) is a compact metric space then it is complete.
This is true by Theorem 3.11(b).

Definition **3.13** $\{s_n\} \subset \mathbb{R}$.
$\{s_n\}$ is *monotone increasing* if $s_n \leq s_{n+1}$ $\forall\ n \in \mathbb{N}$.
$\{s_n\}$ is *monotone decreasing* if $s_n \geq s_{n+1}$ $\forall\ n \in \mathbb{N}$.

Theorem **3.14**
(a) If $\{s_n\} \subset \mathbb{R}$ is increasing and bounded from above, it is convergent.
(b) If $\{s_n\} \subset \mathbb{R}$ is decreasing and bounded from below, it is convergent.

Proof:
(a) Let $s = \sup \{s_n\}$. Then $\forall\ \varepsilon > 0\ \exists\ N_\varepsilon \in \mathbb{N}$ such that $s - \varepsilon < s_{N_\varepsilon} \leq s$.
$\forall\ n > N_\varepsilon$ we have $s - \varepsilon < s_{N_\varepsilon} \leq s_n \leq s < s + \varepsilon$.
Then $|s_n - s| < \varepsilon$, $\forall\ n \in \mathbb{N}$.
(b) Proof is similar.