Content:

Definition	3.21 Partial sums, series, convergent series.
Theorem	3.22 $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \forall\, \varepsilon > 0, \exists\, N_\varepsilon \in \mathbb{N}$ such that $\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$ if $n \ge m > N_\varepsilon$.
Theorem	3.23 If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
Theorem	3.24 Let $\sum_{n=1}^{\infty} a_n$ be a series with non-negative terms (i.e. $a_n \ge 0\ \forall\, n \in \mathbb{N}$). Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \{s_k\}$ is bounded.
Definition	(p. 71) Absolute convergence
Theorem	3.45 If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
Theorem	3.25 Comparison Test (a) If $\lvert a_n\rvert \le c_n\ \forall\, n \ge N_0$ and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too. (b) If $a_n \ge d_n \ge 0\ \forall\, n \ge N_0$ and $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.
Theorem	3.26 Geometric Series $\sum_{n=1}^{\infty} x^n$ converges $\Leftrightarrow -1 < x < 1$.

Homework Week #11, Exercise #6

Discussion $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$ Then $\exists\, N_{\varepsilon_0} \in \mathbb{N}$ such that $\frac{x_{n+1}}{x_n} < L + \varepsilon_0 = r,\ \forall\, n \ge N_{\varepsilon_0}$.

So then we have $x_{n+2} < r x_{n+1} < r^2 x_n$. Let $N_0 = N_{\varepsilon_0} + 1$. Then, $x_{N_0 + k} < r^k x_{N_0}$.
This gives us that $0 \le \lim_{n\to\infty} x_n \le \lim_{k\to\infty} r^k x_{N_0} = x_{N_0} \lim_{k\to\infty} r^k = x_{N_0} \bullet 0 = 0$.

Squeeze Theorem 3.20
If we have $x_n \le y_n \le z_n$, then $\lvert y_n - \alpha\rvert \le \max\{\lvert x_n - \alpha\rvert, \lvert z_n - \alpha\rvert\}$.

Exercise #1 $\Rightarrow$ Exercise #2 $\Rightarrow$ Exercise #3.

Series

Definition **3.21** We need a sequence $\{a_n\}_{n\in\mathbb{N}}$, a sequence of *partial sums*, s_k, $s_k = \sum_{n=1}^{k} a_n$. A pair of 2 sequences $\{\{a_n\}, \{s_k\}\}$ forms a *series*, denoted $\sum_{n=1}^{\infty} a_n$.

Definition A series $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence of partial sums $\{s_k\}$ is convergent. If $s_k \to \alpha$, then $\sum_{k=1}^{\infty} a_k = \alpha$.

Theorem **3.22** $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \forall\ \varepsilon > 0, \exists\ N_\varepsilon \in \mathbb{N}$ such that $\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$ if $n \geq m > N_\varepsilon$.

Proof:

Rewrite the theorem. $\left\{\sum_{n=1}^{k} a_n\right\}_{k \in N} = \{s_k\}$ converges $\Leftrightarrow$ it is Cauchy.

$\left|\sum_{k=n}^{\infty} a_k\right| = |s_m - s_n|$

$s_m = a_1 + a_2 + \cdots + a_{n-1} + a_n + \cdots + a_m$. Note that $a_n + \cdots + a_m = \sum_{k=n}^{\infty} a_k$.

Theorem **3.23** If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof:

This follows from 3.22 above.

$\forall\ \varepsilon > 0, \exists\ N_\varepsilon \in \mathbb{N}$ such that $n = m > N_\varepsilon$ we have $|a_n| < \varepsilon$

$\therefore\ |a_n| \to 0$, hence $a_n \to 0$ by exercise 8, Week 11.

Remark $a_n = (-1)^n$. $|a_n| \to 1$, but a_n diverges.

Note $\lim_{n\to\infty} a_n = 0$ does not imply $\sum_{n=1}^{\infty} a_n$ converges.

$a_n = \frac{1}{n} \to 0$, however $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem **3.24** Let $\sum_{n=1}^{\infty} a_n$ be a series with non-negative terms (i.e. $a_n \geq 0\ \forall\ n \in \mathbb{N}$).

Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \{s_k\}$ is bounded.

Proof:

The fact that $a_n \geq 0\ \forall\ n \in \mathbb{N} \Rightarrow \{s_k\}$ is increasing. $s_{k+1} = s_k + a_{k+1}$.

From Theorem 3.14 (*Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges $\Leftrightarrow$ it is bounded.*) the result holds.

Note The result holds also if all terms are negative, then $\{s_k\}$ is bounded from below.

Definition (p. 71) $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem **3.45** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof:

By Thm 3.22 $\left|\sum_{k=n}^{m} |a_k|\right| < \varepsilon$ so $\sum_{k=n}^{m} |a_k| < \varepsilon$.

So by the triangle inequality we have $\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k| < \varepsilon$.

Remarks

- 3.46 If $a_n \ge 0\ \forall\ n \in \mathbb{N}$ then absolutely convergent is equivalent to convergent.
- Convergent does not imply absolutely convergent.

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is convergent, but $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem **3.25** Comparison Test

(a) If $|a_n| \le c_n\ \forall\ n \ge N_0$ and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $a_n \ge d_n \ge 0\ \forall\ n \ge N_0$ and $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

Proof:

(a) $|a_n| \le c_n\ \forall\ n \ge N_0$ and $\sum_{n=1}^{\infty} c_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$\sum_{n=1}^{\infty} |a_n|$ is a series with positive terms.

$\bar{s}_k = |a_1| + |a_2| + \cdots + |a_k| \le c_1 + c_2 + \cdots + c_k = t_k \le M\ \forall\ k \in \mathbb{N}$.

$\therefore\ \sum_{n=1}^{\infty} a_n$ is convergent.

Without loss of generality, suppose $N_0 = 1$.

(b) This follows from (a) as $a_n \ge d_n \ge 0 \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} |a_n|$ converges, hence, so must $\sum_{n=1}^{\infty} d_n$.

Theorem **3.26** Geometric Series

$\sum_{n=1}^{\infty} x^n$ converges $\Leftrightarrow -1 < x < 1$.

Proof:

$s_k = 1 + x + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}$. $c_k = x^{k+1}$ converges for $-1 < x < 1$.

If x ≥ 1, we use the comparison test with $\sum_{k=0}^{\infty} 1^k$ which diverges.

If $-1 < x < 1$, then $x^{k+1} \to 0$. So $\lim_{k\to\infty} s_k = \frac{1}{1-x}$.

$\sum_{n=0}^{\infty} x_n = \frac{1}{1-x} \Leftrightarrow -1 < x < 1$.