Content:

Theorem 3.37 For any sequence $\{c_n\}$ of positive numbers

$$\liminf_{n\to\infty}\frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty}\sqrt[n]{c_n} \le \limsup_{n\to\infty}\sqrt[n]{c_n} \le \limsup_{n\to\infty}\frac{c_{n+1}}{c_n}.$$

Definition 3.38 Power series

Theorem 3.39 For the power series $\sum_{n=0}^{\infty} c_n z^n$, let $\alpha = \limsup_{n\to\infty}\sqrt[n]{|c_n|}$ and $R = 1/\alpha$. (If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$)

Then $\sum_{n=0}^{\infty} c_n z^n$ converges for all $z \in \mathbb{C}$ with $|z| < R$.

Theorem **3.33** Root Test Given $\sum_{n=1}^{\infty} a_n$ and denote $\alpha = \limsup_{n\to\infty}\sqrt[n]{|a_n|}$.

(a) If $\alpha < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent;

(b) If $\alpha > 1$, $\sum_{n=1}^{\infty} a_n$ is divergent; **(c)** If $\alpha = 1$, we don't know.

Theorem **3.34** Ratio Test **(a)** If $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(b) If $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ for all $n \ge n_0$ where n_0 is a fixed integer, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example For $\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{2^n} + \frac{1}{3^n} + \cdots$ we apply the root test to the subsequences $\{a_{2n-1}\}$ and $\{a_{2n}\}$.

$$\sqrt[2n-1]{\frac{1}{2^n}} = \left(\frac{1}{2}\right)^{\frac{n}{2n-1}} \to \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \text{ since } \lim_{n\to\infty}\frac{n}{2n-1} = 1/2.$$

$$\sqrt[2n]{\frac{1}{3^n}} = \left(\frac{1}{3}\right)^{\frac{n}{2n}} \to \left(\frac{1}{3}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{3}}.$$

So then $\limsup_{n\to\infty}\sqrt[n]{|a_n|} = \frac{1}{\sqrt{2}} < 1$. Thus the series converges.

Note that this series would not satisfy the conditions of the ratio test.

For $\left|\frac{a_{2n}}{a_{2n-1}}\right| = \frac{\frac{1}{3^n}}{\frac{1}{2^n}} = \left(\frac{2}{3}\right)^n \to 0$ and $\left|\frac{a_{2n+1}}{a_{2n}}\right| = \frac{\frac{1}{2^{n+1}}}{\frac{1}{3^n}} = \left(\frac{3}{2}\right)^n \to +\infty$, hence

$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = +\infty$. Clearly (a) is not satisfied. And (b) is not satisfied as the divergence must be true for *all* $n \ge n_0$.

Theorem **3.37** For any sequence $\{c_n\}$ of positive numbers

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof:

We will show $\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n}$. Let $\gamma = \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$.

Then $\gamma \ge 0$.

Case 1: $\gamma > 0$.
Fix an arbitrary $\varepsilon > 0$.

Then $\exists\, N_\varepsilon$ such that $n \ge N_\varepsilon \Rightarrow \gamma - \varepsilon < \frac{c_{n+1}}{c_n}$. Thus $c_n(\gamma - \varepsilon) < c_{n+1}$

$\Rightarrow c_{N_\varepsilon}(\gamma - \varepsilon)^k < c_{N_\varepsilon+k}$

$\Rightarrow c_{N_\varepsilon}^{\frac{1}{N_\varepsilon+k}} (\gamma-\varepsilon)^{\frac{k}{N_\varepsilon+k}} < \left(c_{N_\varepsilon+k}\right)^{\frac{1}{N_\varepsilon+k}}$

$\Rightarrow 1 \bullet (\gamma-\varepsilon) = \lim_{k\to\infty} c_{N_\varepsilon}^{\frac{1}{N_\varepsilon+k}} (\gamma-\varepsilon)^{\frac{k}{N_\varepsilon+k}} < \lim_{k\to\infty} \left(c_{N_\varepsilon+k}\right)^{\frac{1}{N_\varepsilon+k}}$.

Hence, $(\gamma - \varepsilon) < \liminf_{n\to\infty} \sqrt[n]{c_n}$. And as $\varepsilon \to 0$, $\gamma < \liminf_{n\to\infty} \sqrt[n]{c_n}$.

Case 2: $\gamma = 0$.

Then as $c_n > 0$ for all n, we have $\gamma = 0 < \liminf_{n\to\infty} \sqrt[n]{c_n}$.

Remark Divergence by the ratio test $\Rightarrow$ divergence by the root test.
And convergence by the ratio test $\Rightarrow$ convergence by the root test.

The Duhamel test, $\alpha = \lim_{n\to\infty}\left(n\left[1 - \frac{u_{n+1}}{u_n}\right]\right)$ and $\alpha < 1 \Rightarrow$ convergence,

$\alpha > 1 \Rightarrow$ divergence and $\alpha = 1 \Rightarrow$ no information,
is stronger than both tests.

Definition **3.38** The series $\sum_{n=0}^{\infty} c_n z^n$ where $\{c_n\}$ is a sequence of complex numbers (the coefficients) and $z \in \mathbb{C}$ (the variable) is called a *power series*.

Question How can we find a domain $D \subset \mathbb{C}$ such that $\sum_{n=0}^{\infty} c_n z^n$ converges for all $z \in D$?

Theorem **3.39** For the power series $\sum_{n=0}^{\infty} c_n z^n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and $R = 1/\alpha$.
(If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$)

Then $\sum_{n=0}^{\infty} c_n z^n$ converges for all $z \in \mathbb{C}$ with $|z| < R$.

Proof:

$$\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n||z^n|} = \limsup_{n\to\infty} |z|\sqrt[n]{|c_n|} = |z|\alpha.$$

And $|z|\alpha < 1$ if $|z| < R$.