Content:

Definitions 3.1 Sequence, bounded, convergent, divergent

Theorem 3.2 Let (X, d) be a metric space, let $\{p_n\}$ be a sequence in X.
(a) $p_n \to p \Leftrightarrow$ for every neighborhood of p just finitely many terms of $\{p_n\}$ stay outside of that neighborhood.
(b) If $p_n \to p$ and $p_n \to p'$, then $p = p'$.
(c) If $p_n \to p$ then $\{p_n\}$ is bounded.
(d) If $E \subset X$ and p is a limit point of E, then $\exists \{p_n\} \subset E$ such that $p_n \to p$.

Theorem 3.3 Let $X = \mathbb{C}$. Suppose $s_n \to s$ and $t_n \to t$. Then
(a) $s_n + t_n \to s + t$; (b) $c \bullet s_n \to c \bullet s \; \forall \, c \in \mathbb{C}$;
(c) $s_n \bullet t_n \to s \bullet t$; (d) $t_n/s_n \to t/s$ (if $s_n \neq 0 \; \forall \, n \in \mathbb{N}$, and $s \neq 0$)

Chapter 3

Definition A *sequence* is a function $p: \mathbb{N} \to X$ where X is a metric space.
We denote $p_1 = p(1)$, p_2, $= p(2)$, ..., $p_n = p(n)$, ...
And we denote a sequence as $\{p_n\}_{n \in \mathbb{N}}$.

Examples
1. Fix $x_0 \in X$. Define $p_n = x_0 \; \forall \, n \in \mathbb{N}$. This is the constant sequence.
2. Let $p_n = 1/n$, $n \in \mathbb{N}$.
3. Let $p_n = 2^n$, $n \in \mathbb{N}$.
4. Let $X = \mathscr{C}[0, 1]$, the set of continuous functions $f: [0, 1] \to \mathbb{R}$.
Define $d(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$.
Let $p_n(x) = x^n \; \forall \, n \in \mathbb{N}$. Then $p_1(x) = x$, $p_2(x) = x^2$, ..., $p_n(x) = x^n$.

Definition A sequence $\{p_n\}$ is *bounded* if $\exists \, x_0 \in X$ and $\exists \, M > 0$ such that $d(x_0, p_n) < M \; \forall \, n \in \mathbb{N}$.

Definition A sequence $\{p_n\}$ *converges* to $p \in X$ if $\forall \, \varepsilon > 0$, $\exists \, N_\varepsilon \in \mathbb{N}$ such that $d(p, p_n) < \varepsilon$, $\forall \, n \in \mathbb{N}$.
Denote $\lim_{n \to \infty} p_n = p$ or $p_n \to p$. We say $\{p_n\}$ is *convergent*.

Definition If a sequence is not convergent to any $p \in X$, it is *divergent*.

Example By the definition above, the following sequence is divergent:
Let $X = \mathbb{Q}$. Let $p_n = \sum_{k=0}^{n} \frac{1}{k!}$. $p_1 = 1$
$p_2 = 1 + 1 = 2$
$p_3 = 1 + 1 + 1/2 = 2.5$
$p_4 = 1 + 1 + 1/2 + 1/6 = 2.66...$
$p_n \to e$.
This diverges in $\mathbb{Q}$, but is converges to a hole around e in the space.

Examples 1. $(1/n) \to 0$.

2. $p_n = 1 + (-1)^n/n \to 1$. This is not monotonic convergent.
It jumps around. $p_1 = 1 + -1/1 = 0$
$p_2 = 1 + 1/2 = 3/2$
$p_3 = 1 + -1/3 = 2/3$
$p_4 = 1 + 1/4 = 5/4$

3. $p_n = q^n$, $n \in \mathbb{N}$ and $-1 < q \le 1$. $\{p_n\}$ is convergent.

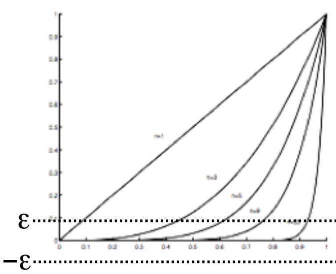
$$p_n \to p = \begin{cases} 0 \text{ if } -1 < q < 1 \\ 1 \text{ if } q = 1 \end{cases}$$

4. Let $p_n = (-1)^n$. Then $\{p_n\} = \{-1, 1, -1, 1, ...\}$.
This is an alternating sequence.
It contains 2 subsequences that converge differently to -1 and 1.

5. Let $p_n = (-2)^n$.
Then $\{p_n\}$ has 2 subsequences that diverge differently to ∞ and $-\infty$.

6. $p_n = 2^n$. Then $\{p_n\}$ diverges only to ∞.

7. Let $X = \mathscr{C}[0, 1]$, the set of continuous functions
$f: [0, 1] \to \mathbb{R}$.
Let $p_n(x) = x^n$ $\forall$ $n \in \mathbb{N}$.
Then $x^n \to \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } x = 1 \end{cases}$.
This is not uniformly continuous.
Even if we consider $X = B[0, 1]$
where $d(f, g) = \sup_{0 \le x \le 1} |f(x) - g(x)|$
this sequence still does not converge.



Note that in this metric space a "ball" looks like a strip about a given function value.

Theorem **3.2** Let (X, d) be a metric space, let $\{p_n\}$ be a sequence in X.
(a) $p_n \to p \Leftrightarrow$ for every neighborhood of p just finitely many terms of $\{p_n\}$ stay outside of that neighborhood.
(b) If $p_n \to p$ and $p_n \to p'$, then $p = p'$.
(c) If $p_n \to p$ then $\{p_n\}$ is bounded.
(d) If $E \subset X$ and p is a limit point of E, then $\exists \{p_n\} \subset E$ such that $p_n \to p$.

Proof:
(a) This is simply true by definition.

(b) We want to show $\forall\ \varepsilon > 0\ d(p, p') < \varepsilon$.
Let $\varepsilon > 0$. Then $\exists\ N_\varepsilon \in \mathbb{N}$ such that $d(p, p_n) < \varepsilon/2\ \forall\ n > N_\varepsilon$ and $\exists\ N'_\varepsilon \in \mathbb{N}$ such that $d(p', p_n) < \varepsilon/2\ \forall\ n > N'_\varepsilon$. Let $N'' = \max\{N_\varepsilon, N'_\varepsilon\}$.
Then $d(p, p') \le d(p, p_{N''}) + d(p_{N''}, p') < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

(c) Let $\varepsilon = 1$. Then $\exists\ N_1 \in \mathbb{N}$ such that $d(p, p_n) < 1\ \forall\ n \in \mathbb{N}$.
Only finitely many terms of $\{p_n\}$ lie outside this neighborhood, $p_1, p_2, \ldots, p_{N_1}$. Let $M = \max\{1, d(p, p_1), d(p, p_2), \ldots, d(p, p_{N_1})\}$.
Then $d(p, p_n) < M\ \forall\ n \in \mathbb{N}$.

(d) Let $\varepsilon = 1/n$. Since p is a limit point, then $[B_{1/n}(p)\backslash\{p\}] \cap E \neq \varnothing$.
$\Rightarrow \forall\ n \in \mathbb{N}\ \exists\ p_n \in B_{1/n}(p) \cap E \Rightarrow d(p, p_n) < 1/n \Rightarrow p_n \to p$.

Note (d) If $p \in E$, then we can just choose $p_n = p\ \forall\ n \in \mathbb{N}$. If $p \notin E$, then we must create $\{p_n\}$ where $p_n \in E$ for each n.

Theorem **3.3** Let $X = \mathbb{C}$. Then $d(a + ib, c + id) = \sqrt{(a-c)^2 + (b-d)^2}$
Suppose $s_n \to s$ and $t_n \to t$. Then
(a) $s_n + t_n \to s + t$; **(b)** $c \bullet s_n \to c \bullet s \ \forall\, c \in \mathbb{C}$;
(c) $s_n \bullet t_n \to s \bullet t$; **(d)** $t_n/s_n \to t/s$ (if $s_n \neq 0 \ \forall\, n \in \mathbb{N}$, and $s \neq 0$)

Proof:
(a) Let $\varepsilon > 0$. $s_n \to s \Rightarrow \exists\, N_{\varepsilon/2} \in \mathbb{N}$ such that $d(s_n, s) < \varepsilon/2 \ \forall\, n > N_{\varepsilon/2}$
and $t_n \to t \Rightarrow \exists\, N'_{\varepsilon/2}$ such that $d(t_n, t) < \varepsilon/2 \ \forall\, n > N'_{\varepsilon/2}$.
Note that $d(a + ib + x + iy, c + id + x + iy) =$
$\sqrt{(a+x-(c+x))^2 + (b+x-(d+x))^2} = \sqrt{(a-c)^2 + (b-d)^2}$.

So then
$$\begin{aligned} d(s_n + t_n, s + t) &\leq d(s_n + t_n, s_n + t) + (s_n + t, s + t) \\ &= d(t_n, t) + d(s_n, s) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

$\therefore\ s_n + t_n \to s + t$.
(b) If $c = 0$, then the result is true.
If $c \neq 0$, then let $\varepsilon > 0$.
Then $s_n \to s \Rightarrow \exists\, N_{\varepsilon/|c|} \in \mathbb{N}$ such that $d(s_n, s) < \varepsilon/|c| \ \forall\, n > N_{\varepsilon/|c|}$.
And $d(cs_n, cs) = |c|d(s_n, s) < |c| \bullet \varepsilon/|c| = \varepsilon$. $\therefore\ c \bullet s_n \to c \bullet s$.
(c) Note that by Thm 3.2 (c) $\{s_n\}$ is bounded by, say M.
Then $d(s_n, 0) < M \ \forall\, n \in \mathbb{N}$. Or equivalently, $|s_n| < M \ \forall\, n \in \mathbb{N}$.
If $t \neq 0$, $s_n \to s \Rightarrow \exists\, N_{\varepsilon/(2|t|)} \in \mathbb{N}$ such that $d(s_n, s) < \varepsilon/(2|t|) \ \forall\, n > N_{\varepsilon/(2|t|)}$.
If $t = 0$, then we leave it alone as $|t||s_n - s| = 0 < \varepsilon$.
$t_n \to t \Rightarrow \exists\, N'_{\varepsilon/(2M)}$ such that $d(t_n, t) < \varepsilon/2 \ \forall\, n > N'_{\varepsilon/(2M)}$.

$$\begin{aligned} d(s_n \bullet t_n, s \bullet t) &\leq d(s_n \bullet t_n, s_n \bullet t) + d(s_n \bullet t, s \bullet t) \\ &= |s_n t_n - s_n t| + |s_n t - st| \\ &= |s_n||t_n - t| + |t||s_n - s| \\ &< M \bullet \varepsilon/2M + |t| \bullet \varepsilon/2|t| = \varepsilon. \end{aligned}$$

$\therefore\ s_n \bullet t_n \to s \bullet t$.
(d) We will show $1/s_n \to 1/s$. Thus, applying part (c) we will have $t_n/s_n \to t/s$.

We want to show $d(1/s_n, 1/s) = \dfrac{|s - s_n|}{|s_n||s|} < \varepsilon$. Let $\delta = |s|/3$.

Then at most $s_1, s_2, \ldots, s_{N_\delta}$ lie outside the δ neighborhood of s.
Since $s_n \neq 0 \ \forall\, n \in \mathbb{N}$ and $s \neq 0$, then $\exists\, r > 0$ such that $B_r(0) \cap (\{s_n\} \cup \{s\}) = \varnothing$.
We can choose $r = \min\{(2|s|)/3, |s_1|, |s_2|, \ldots, |s_{N_\delta}|\}$.
$\forall\, n > N_\delta$, $|s - s_n| < |s|/3 \Rightarrow |s - 0| \leq |s - s_n| + |s_n - 0| < |s|/3 + |s_n|$
$\Rightarrow r \leq (2|s|/3) < |s_n|$. We also have that $r \leq (2|s|/3) < |s|$.

Thus $1/|s_n| < 1/r$, hence $\dfrac{|s - s_n|}{|s_n||s|} < \dfrac{|s - s_n|}{r^2}$.

So if we choose $N_{\varepsilon'}$ such that $|s - s_n| < \varepsilon r^2$, $\forall\, n > N_{\varepsilon'}$, we then have
$\dfrac{|s - s_n|}{|s_n||s|} < \varepsilon$, as desired.