Content:

Theorem 3.50 Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, and $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the product of the series $\sum_{n=1}^{\infty} c_n$ as defined above is convergent and $\sum_{n=1}^{\infty} c_n = AB$.

Definition 3.52 Rearrangements

Theorem 3.55 If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then every rearrangement is convergent to the same sum.

Theorem **3.50** Suppose $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, and $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then the product of the series $\sum_{n=0}^{\infty} c_n$ as defined above is convergent and $\sum_{n=0}^{\infty} c_n = AB$.

Proof:

Let $\sum_{k=0}^{n} a_k = A_n$, $\sum_{k=0}^{n} b_k = B_n$, $\sum_{k=0}^{n} c_k = C_n$, and $\beta_n = B_n - B$.

So then $B_n = B + \beta_n$.

And C_n

$= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)$

$= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_nb_0.$

$= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0$

$= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0)$

$= B(a_0 + a_1 + \cdots + a_n) + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0$

$= BA_n + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0$

We want to show $a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0$ drops to 0.

Since $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, we can let $M = \sum_{n=0}^{\infty} |a_n|$.

Let $\varepsilon > 0$. Since $\sum_{n=0}^{\infty} b_n = B$, $\exists N \in \mathbb{N}$ such that $|B_n - B| = |\beta_n| < \varepsilon/M$ if $n \geq N$.

Let $n > N$.

Let $\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0$

$= a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-N}\beta_N + a_{n-N+1}\beta_{N-1} + \cdots + a_n\beta_0$

$\leq a_0(\varepsilon/M) + a_1(\varepsilon/M) + \cdots + a_{n-N}(\varepsilon/M) + a_{n-N+1}\beta_{N-1} + \cdots + a_n\beta_0.$

Then $|\gamma_n| \leq (\varepsilon/M)\left(\sum_{k=0}^{n-N}|a_k|\right) + |a_{n-N+1}||\beta_{N-1}| + \cdots + |a_n||\beta_0|$

Since $\sum_{n=0}^{\infty} a_n$ is convergent $\Rightarrow a_n \to 0$, then

$\limsup_{n\to\infty}|\gamma_n| \leq \varepsilon + 0 \bullet \beta_{N-1} + \cdots + 0 \bullet \beta_0 = \varepsilon.$

Since $0 \leq \liminf_{n\to\infty}|\gamma_n| \leq \limsup_{n\to\infty}|\gamma_n| \leq \varepsilon$, then $\liminf_{n\to\infty}|\gamma_n| = \limsup_{n\to\infty}|\gamma_n| = 0$,

hence $\lim_{n\to\infty}\gamma_n = 0$.

So then $\lim_{n\to\infty} C_n = \lim_{n\to\infty}(BA_n + \gamma_n) = AB$, then $\sum_{n=0}^{\infty} c_n = AB$.

Rearrangements

Definition **3.52** Given $\sum_{n=0}^{\infty} a_n$, define $\sum_{m=1}^{\infty} a_{k_m}$ where $k: \mathbb{N} \to \mathbb{N}$ is a bijection with $k(m) = k_m$. We say $\sum_{m=1}^{\infty} a_{k_m}$ is a *rearrangement* of $\sum_{n=0}^{\infty} a_n$.

Example

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots$$
$$= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots.$$

Theorem **3.55** If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then every rearrangement is convergent to the same sum.

Proof:

Let $\varepsilon > 0$. $\exists\, N \in \mathbb{N}$ such that $\sum_{i=n}^{m} |a_i| < \varepsilon$ for $N \le n \le m$.

Let $\sum_{m=1}^{\infty} a_{k_m}$ be a rearrangement of $\sum_{n=0}^{\infty} a_n$.

We can find $L \in \mathbb{N}$ such that $\{1, 2, ..., N\} \subset \{k_1, k_2, ..., k_{L-1}\}$.

Then for each j $(1 \le j \le N)$, $a_j = a_{k_i}$ for some $i \in \{1, 2, ..., L-1\}$.

Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{m=1}^{n} a_{k_m}$. For $n \ge \max\{N, K_L\}$,

$|s_n - t_n| \le \sum_{i=N}^{n} |a_i|$. So then if $s_n \to s$, we have $t_n \to s$.

Note If $\sum_{n=0}^{\infty} a_n$ is conditionally convergent, the result does not hold.

Let $\sum_{n=0}^{\infty} a_n$ be conditionally convergent.

Let $\{p_m\}$ be the positive terms of $\{a_n\}$ and $\{q_k\}$ be the negative terms.

Since $\sum_{n=0}^{\infty} a_n$ is convergent, $\sum_{m=0}^{\infty} p_m$ and $\sum_{k=0}^{\infty} q_k$ both diverge while $p_m \to 0$ and $q_k \to 0$.

Let $x \in \mathbb{R}$. Then $\exists\, \{p_1, p_2, ..., p_{k1}\}$ where $k1$ is the largest index such that $p_1 + p_2 + ... + p_{k1} > x$.

And $\exists\, \{q_1, q_2, ..., q_{k2}\}$ where $k2$ is the largest index such that

$p_1 + p_2 + ... + p_{k1} + q_1 + q_2 + ... + q_{k2} < x$.

We can continue forming sums that satisfy these conditions infinitely.

Let $s_{k2} = p_1 + p_2 + ... + p_{k1} + q_1 + q_2 + ... + q_{k2}$.

Then $|s_{k2} - x| < |q_{k2}|$. Since $q_k \to 0$, then

this rearrangement forces $\sum_{n=0}^{\infty} a_n$ to converge to x.

Homework Week 13, Exercise 9.

Discussion Let $a_n = \begin{cases} \frac{1}{n^2} \text{ if } n \neq 2^k \\ 1 \text{ if } n = 2^k \end{cases}$, where $k \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} \frac{1}{1+na_n}$ converges.

Let $\{b_m\} = \{a_{2^k}\}$ and $\{c_r\} = \{a_n\} \setminus \{b_m\}$ maintaining the order of appearance.

Then $\sum_{m=1}^{\infty} b_m = \sum_{k=1}^{\infty} \frac{1}{1+2^k}$ which is clearly convergent. And

$\sum_{r=0}^{\infty} c_r = \sum_{n \neq 2^k} \frac{\frac{1}{n^2}}{1+\frac{1}{n^2}} = \sum_{n \neq 2^k} \frac{1}{n^2+n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+n} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ which also converges. Thus $\sum_{m=1}^{\infty} b_m + \sum_{r=0}^{\infty} c_r$ is convergent.

Since $\sum_{n=1}^{\infty} \frac{1}{1+na_n} = \sum_{n=1}^{\infty} \left| \frac{1}{1+na_n} \right| = \sum_{m=1}^{\infty} b_m + \sum_{r=0}^{\infty} c_r$ a convergent rearrangement of $\sum_{n=1}^{\infty} \frac{1}{1+na_n}$, then every rearrangement converges to the same sum.