Young's Inequaltiy

$ab \le a^p/p + b^q/q$

where $1/p + 1/q = 1$ and $a, b > 0$.

Proof:
$\ln(a^p/q + b^q/q)$

$\ln(ab) = \ln(a) + \ln(b)$

$= (1/p)\ln a^p + (1/q)\ln b^q \le \ln((1/q)a^p + (1/p)b^q)$

by the inequalty

$\lambda f(a) + (1 - \lambda) f(b) \le f(\lambda a + (1-\lambda) b)$.

Taking ln of both sides we have

$ab \le a^p/p + b^q/q$, as desired.

Hölder's Inequality

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \bullet \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

where $1/p + 1/q = 1$.

Proof:
By Young's Inequality, we have

$\dfrac{|x_i y_i|}{|\mathbf{x}|_p |\mathbf{y}|_q} \le \dfrac{1}{p}\left(\dfrac{|x_i|}{|\mathbf{x}|_p}\right)^p + \dfrac{1}{q}\left(\dfrac{|y_i|}{|\mathbf{y}|_q}\right)^q$. So then

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{|\mathbf{x}|_p |\mathbf{y}|_q} \le \sum_{i=1}^{n} \frac{1}{p}\left(\frac{|x_i|}{|\mathbf{x}|_p}\right)^p + \sum_{i=1}^{n} \frac{1}{q}\left(\frac{|y_i|}{|\mathbf{y}|_q}\right)^q.$$

Thus

$$\frac{1}{|\mathbf{x}|_p |\mathbf{y}|_q} \sum_{i=1}^{n} |x_i y_i| \le \frac{1}{p} \bullet \left(\frac{1}{|\mathbf{x}|_p}\right)^p \sum_{i=1}^{n} |x_i|^p + \frac{1}{q} \bullet \left(\frac{1}{|\mathbf{y}|_q}\right)^q \sum_{i=1}^{n} |y_i|^q$$

$$= \frac{1}{p} \bullet \left(\frac{1}{|\mathbf{x}|_p}\right)^p \left(|\mathbf{x}|_p\right)^p + \frac{1}{q} \bullet \left(\frac{1}{|\mathbf{y}|_q}\right)^q \left(|\mathbf{y}|_q\right)^q$$

$= \dfrac{1}{p} \bullet 1 + \dfrac{1}{q} \bullet 1 = 1$. So then

$$\sum_{i=1}^{n} |x_i y_i| \le |\mathbf{x}|_p |\mathbf{y}|_q = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \bullet \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}.$$

Cauchy-Schwartz's Inequality

$$\left|\sum_{i=1}^{n} x_i \bar{y}_i\right|^2 \le \sum_{i=1}^{n} |x_i|^2 \bullet \sum_{i=1}^{n} |y_i|^2$$

Proof:
Use Hölder's Inequality with
$p = q = 2$, to get

$$\sum_{i=1}^{n} |x_i \bar{y}_i| \le \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \bullet \left(\sum_{i=1}^{n} |\bar{y}_i|^2\right)^{1/2}$$

So then $\left(\sum_{i=1}^{n} |x_i \bar{y}_i|\right)^2 \le \sum_{i=1}^{n} |x_i|^2 \bullet \sum_{i=1}^{n} |\bar{y}_i|^2$

And by the triangle inequality

$\left|\sum_{i=1}^{n} x_i \bar{y}_i\right| \le \left|\sum_{i=1}^{n} |x_i \bar{y}_i|\right|$. Thus

$$\left|\sum_{i=1}^{n} x_i \bar{y}_i\right|^2 \le \sum_{i=1}^{n} |x_i|^2 \bullet \sum_{i=1}^{n} |\bar{y}_i|^2$$

$$= \sum_{i=1}^{n} |x_i|^2 \bullet \sum_{i=1}^{n} |y_i|^2.$$

Minkowski's Inequality

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

Proof:

$$\sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n} |x_i + y_i| |x_i + y_i|^{p-1}$$

$$\le \sum_{i=1}^{n} (|x_i| + |y_i|) |x_i + y_i|^{p-1}$$

by the triangle inequality

$$= \sum_{i=1}^{n} |x_i| |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| |x_i + y_i|^{p-1}$$

$$\le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \bullet \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{1/q} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p} \bullet \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{1/q}$$

by the Hölder Inequality.

$$\sum_{i=1}^{n} |x_i + y_i|^p \le$$

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/q} \left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}\right) \le$$

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/q} \left(|\mathbf{x}|_p + |\mathbf{y}|_p\right).$$

So then $|\mathbf{x} + \mathbf{y}|_p = \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} =$

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1-1/q} \le |\mathbf{x}|_p + |\mathbf{y}|_p.$$