Week 2, Practice Problems, Assigned Friday, February 4, 2011

1. Read Theorem 4.20 and its proof.
2. Read the Discontinuities section on pages 94 and 95.

3. Suppose that $f: [0, +\infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = L$ is a finite number.

Show that f is uniformly continuous.

Proof:
Let $\varepsilon > 0$.
Since $\lim_{x\to\infty} f(x) = L$, then $\exists\, M > 0 \ni \forall\, x \ge M, |f(x) - L| < \varepsilon/2$. So then
$\forall\, x, y \in [M, +\infty), |f(x) - f(y)| \le |f(x) - L| + |L - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
Continuity of $f \Rightarrow \exists\, \delta > 0 \ni \forall\, x, y \in [0, M]$ where $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$
by Theorem 4.19 (*A continuous mapping of a compact set is uniformly continuous*). And
$\forall\, x, y \in (M - \delta, M + \delta)$, we have $|f(x) - f(y)| \le |f(x) - f(M)| + |f(M) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
$\therefore$ f is uniformly continuous.

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4. Let X be a metric space and $x_0 \in X$.
Show that $f: X \to \mathbb{R}, f(x) = d(x, x_0)$ is uniformly continuous.

Proof:

Let $\varepsilon > 0$, let $\delta = \varepsilon$, let $x_0, p, q \in X$.
Since $d(p, x_0) \le d(p, q) + d(x_0, q) \Rightarrow d(p, x_0) - d(x_0, q) \le d(p, q)$ and
$d(x_0, q) \le d(p, q) + d(p, x_0) \Rightarrow d(x_0, q) - d(p, x_0) \le d(p, q)$, then for $d(p, q) < \delta$ we have
$|d(p, x_0) - d(q, x_0)| \le d(x, y) < \varepsilon$.
$\therefore$ f is a uniformly continuous function on X.

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5. Let X, Y be metric spaces and $f: X \to Y$ be uniformly continuous.
Show that if $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.
Give an example showing that this is not true if f is just continuous.

Proof:
Let $\{x_n\}$ be a Cauchy sequence in X and let $\varepsilon > 0$.
By uniform continuity of f, $\exists\, \delta > 0 \ni |x_n - x_m| < \delta \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$.
Since $\{x_n\}$ is Cauchy, then $\exists\, N \in \mathbb{N} \ni \forall\, n, m \ge N, |x_n - x_m| < \delta$.
$\therefore$ $n, m \ge N \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$ which gives us that $\{f(x_n)\}$ is Cauchy.

Example:
Suppose f is just continuous. Consider $f: (0, +\infty) \to \mathbb{R}$ defined by $f(x) = 1/x$.
This is continuous as $\forall\, p \in (0, +\infty), \forall\, \varepsilon > 0$,
$\exists\, \delta = \varepsilon \ni |x - p| < \delta \Rightarrow |f(x) - f(p)| = |(x - p)/(xp)| < |x - p| < \varepsilon$.
Let $\delta > 0$. Choose $n \in \mathbb{N} \ni 1/n < \delta$. Let $\varepsilon = 1$. Let $x_n = 1/n$.
Then $|x_n - x_{n+1}| = |1/n - 1/(n+1) < 1/n = \delta$ and
$|f(x_n) - f(x_{n+1})| = |1/x_n - 1/x_{n+1}| = |n - (n+1)| = 1 \ge \varepsilon$.

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6. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(x)>0, \forall\, x\in\mathbb{R}$ and $\lim_{x\to\pm\infty} f(x)=0$. Show that there exists $x_0\in\mathbb{R}\ni f(x_0)\geq f(x), \forall\, x\in\mathbb{R}$.

Proof:

Let $t\in\mathbb{R}$. Then $f(t)>0$.

$\lim_{x\to\pm\infty} f(x)=0 \Rightarrow \exists\, M\in\mathbb{R}^{+}\ni t\leq M$ and $\forall\, x\in(-\infty,-M]\cup[M,+\infty), f(x)<f(t)$.

Since f is continuous, then $f[-M,M]$ is closed and bounded, hence, by Theorem 4.16

(*For $f\!:\!X\to\mathbb{R}$, X compact metric space and f continuous, $\exists\, p,q\in X\ni f(p)=\sup(f(X))$ & $f(q)=\inf(f(X))$*)

attains a maximum value, call it $f(x_0)$.

Then $f(t)\leq f(x_0)$, hence $\forall\, x\in(-\infty,-M]\cup[M,+\infty), f(x)<f(x_0)$.

$\therefore$ There exists $x_0\in\mathbb{R}\ni f(x_0)\geq f(x), \forall\, x\in\mathbb{R}$.

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7. Solve the following exercises from Rudin's Text, p. 98 - 102: #1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 14, 20, 21.

Rudin, Ch 4, p. 98 #1. Suppose f is a real function defined on R^1 which satisfies $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ for every $x \in R^1$. Does this imply that f is continuous?

> No. If $f(0) = 1$ and $f(x) = 0$ for all $x \neq 0$, then $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ but f is not continuous.

Proof:
Let $\varepsilon = 1/2$. Then
$\forall\ \delta > 0, \exists\ x_0 = (1/2)\delta$ such that $|x_0 - 0| < \delta \Rightarrow |f(x_0) - f(0)| = |0 - 1| = 1 > 1/2$.
$\therefore f$ is not continuous.

Rudin, Ch 4, p. 98 #2. If f is a continuous mapping of a metric space X into a metric space Y, prove that $f(\overline{E}) \subset \overline{f(E)}$ for every set $E \subset X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof:
Let $y \in f(\overline{E})$. Then $\exists\ x \in \overline{E}$ such that $f(x) = y$.
Let $\varepsilon > 0$. Then by continuity of f, $f^{-1}(N_\varepsilon(y))$ is an open neighborhood of x.
Since $x \in \overline{E}$, then $\forall\ \delta > 0, N_\delta(x)^* \cap E \neq \varnothing$.
Let $w \in f^{-1}(N_\varepsilon(y)) \cap E \ni w \neq x$.
Then $f(w) \in f(f^{-1}(N_\varepsilon(y)) \cap E) \subset [f(f^{-1}(N_\varepsilon(y))) \cap f(E)] \subset N_\varepsilon(y) \cap f(E)$.
And $f(w) \neq f(x)$. Thus $N_\varepsilon(y)^* \cap f(E) \neq \varnothing$.
Since $N_\varepsilon(y)$ is an arbitrary neighborhood of y, then $y \in \overline{f(E)}$.

Example for $f(\overline{E}) \neq \overline{f(E)}$:
Let $f(x) = 1/x$, $X = (0, \infty)$, $Y = R$, $E = \mathbb{Z}_+$.
Then $\overline{E} = E$, and $f(\overline{E}) = \{1/n : n \in \mathbb{Z}_+\}$.
But, since $f(E) = \{1/n : n \in \mathbb{Z}_+\}$, then $\overline{f(E)} = \{0\} \cup \{1/n : n \in \mathbb{Z}_+\}$.

Rudin, Ch 4, p. 98 #3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof:

Let $A = X - Z(f)$. We will show that A is open, hence $Z(f)$ is closed.
Let $p \in A$, then $f(p) \neq 0$.
Note that if $|f(x) - f(p)| < |f(p)/2|$, $x \in A$.
Let $\varepsilon = |f(p)/2|$.
Since f is continuous, $\exists\ \delta > 0$ such that $|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$.
Thus $(p - \delta, p + \delta) \subset$ A
$\therefore$ A is open, hence, $Z(f)$ is closed.

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Rudin, Ch 4, p. 98 #4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Proof:
To show $f(E)$ is dense in $f(X)$, let $y \in f(X)$ and note that $\exists\ x_0 \in X$ such that $f(x_0) = y$.
Since E is dense in X, then x_0 is a limit point of E or $x_0 \in E$.
If $x_0 \in E$, then $f(x_0) = y \in f(E)$.
Suppose x_0 is a limit point of E.
Let $\varepsilon > 0$. Since f is continuous, then $\exists\ \delta > 0$ such that $f(B(x_0, \delta)) \subset B(y, \varepsilon)$.
Since x_0 is a limit point of E, $\exists\ p \in B(x_0, \delta) \ni p \neq x_0$.
Hence $f(p) \neq y$ and $f(p) \in B(y, \varepsilon)$.
So then y is a limit point of $f(E)$.
$\therefore f(E)$ is dense in $f(X)$.

To show $g(p) = f(p)$ for all $p \in X$ if $g(p) = f(p)$ for all $p \in E$, choose $p \in X - E$.
Then p is a limit point of E.
Let $\varepsilon > 0$. Since f and g are continuous mappings of X into Y, then
$\exists\ \delta > 0$ such that $f(B(p, \delta)) \subset B(f(p), \varepsilon/4)$ and $g(B(p, \delta)) \subset B(g(p), \varepsilon/4)$.
Choose $N \in \mathbb{N}$ large enough so that $1/N < \delta$
Since $p \in E'$, then for each $n \geq N$, we can choose $q_n \in E \ni q_n \neq p$ and $q_n \in B(p, \delta)$.
By hypothesis, $g(q_n) = f(q_n)$ for each n, thus we have
$d(f(p), g(p)) \leq d(f(p), f(q_n)) + d(f(q_n)), g(q_n)) + d(g(q_n), g(p)) =$
$d(f(p), f(q_n)) + d(g(q_n)), g(q_n)) + d(g(q_n), g(p)) \leq \varepsilon/4 + 0 + \varepsilon/4 < \varepsilon$.
$\therefore\ g(p) = f(p)$ for all $p \in X$.

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Rudin, Ch 4, p. 99 #5. If f is a real continuous function defined on a closed set $E \subset \mathbb{R}^1$, prove that there exist continuous real functions g on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions are called continuous extensions of f from E to $\mathbb{R}^1$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions. Hint: Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chap. 2). The result remains true if $\mathbb{R}^1$ is replace by any metric space, but the proof is not so simple.

Proof:
Let f be a real continuous function defined on a closed set $E \subset \mathbb{R}^1$.
Let $a = \min\{x \mid x \in E\}$ and let $b = \max\{x \mid x \in E\}$.

Define $g(x)\begin{cases} f(b) : b \le x \\ f(x) : a \le x \le b. \\ f(a) : x \le a \end{cases}$

Let $\varepsilon > 0$.
(1) Let $t > b$ and let $\delta = |t - b|$. Then $|x - t| < \delta \Rightarrow |f(x) - f(t)| = |f(b) - f(b)| = 0 < \varepsilon$.
(2) Let $t < a$ and let $\delta = |t - a|$. Then $|x - t| < \delta \Rightarrow |f(x) - f(t)| = |f(a) - f(a)| = 0 < \varepsilon$.
(3) Let $a < t < b$. Then f is continuous by hypothesis.
Let $t = b$. Then by (1) $g(b+) = f(b)$ and by (3) $g(b-) = f(b)$.
Let $t = a$. Then by (2) $g(a-) = f(a)$ and by (3) $g(a+) = f(a)$.
$\therefore$ g is continuous at x $\forall$ $x \in \mathbb{R}$.

If E is open, the result does not hold in general.
Example: Let $E = (-\infty, 0) \cup (0, +\infty)$ and let $f: E \to \mathbb{R}$ be defined by $f(x) = 1/x$.
Suppose $\exists$ g continuous on $\mathbb{R}$ and $g(x) = f(x)$ $\forall$ $x \in E$.
Then $\exists$ $y \in \mathbb{R} \ni g(0) = y$.
Let $\delta > 0$. Then $\exists$ $n \in \mathbb{N} \ni 1/(2ny) < \delta \Rightarrow$
$|g(1/(ny) - g(0)| = |f(1/(2ny)) - y| = |2ny - y| \ge (2n - 1)y$.
$\therefore$ g is not continuous at 0.

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Rudin, Ch 4, p. 99 #6. If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.
Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Proof:
Let $Y = \{(x, f(x)) : x \in E\}$.
$\Rightarrow$:
Assume E is compact and f is continuous on E.
Then $f(E)$ is compact by
Theorem 4.14 (*For X, Y metric spaces, X compact, $f : X \to Y$ is continuous $\Rightarrow f(X)$ is compact.*).
Let $w \in Y'$, then $w = (a, b)$ for some $a, b \in \mathbb{R}$.
Let $\varepsilon > 0$ and choose $n \in \mathbb{N} \ni 1/n < \varepsilon$.
Since $B(w, \varepsilon)^* \cap Y \neq \emptyset$, $\exists\, p_n \in E \ni (p_n, f(p_n)) \in B(w, 1/n)^* \cap Y$ and $(p_n, f(p_n)) \neq w$.
For each n, $d(w, (p_n, f(p_n)) = \sqrt{(a - p_n)^2 + (b - f(p_n))^2} < \varepsilon$.
Since $|a - p_n| \leq \sqrt{(a - p_n)^2 + (b - f(p_n))^2}$ and $|b - f(p_n)| \leq \sqrt{(a - p_n)^2 + (b - f(p_n))^2}$ then $p_n \to a$ and $f(p_n) \to b$. Since E and $f(E)$ are compact, then $a \in E$ and $b \in f(E)$.
$\therefore$ w $\in$ Y, hence Y is closed.
Since $E \times f(E)$ is compact, then Y is bounded.
$\therefore$ Y is compact.

$\Leftarrow$:
Assume E and Y are compact and suppose f is not continuous on E.
Then $\exists\, p \in E$ such that f is not continuous at p.
Let $\delta_n = \dfrac{1}{n}$ for each $n \in \mathbb{N}$.
Construct the sequence $\{p_n\}$ in E such that $|p - p_n| < \delta_n$,
then since f is not continuous on E, for each n, $\exists\, \varepsilon_n > 0 \ni |f(p) - f(p_n)| \geq \varepsilon_n$.
Since $|p - p_n| \to 0$ and E is compact, then $p_n \to p$.
Since $\{(p_n, f(p_n))\}$ is an infinite subset of a compact set Y,
then $\{(p_n, f(p_n))\} \to (p, a)$ where $(p, a) \in Y$.
So $\exists\, N \in \mathbb{N} \ni n \geq N \Rightarrow \sqrt{\left(p_n - p\right)^2 + \left(f(p_n) - a\right)^2} < \varepsilon_n$
and $\sqrt{\left(p_n - p\right)^2 + \left(f(p_n) - f(p)\right)^2} \geq \varepsilon_n$.
Since $p_n \to p$, then $(p_n, f(p_n)) \to (p, a)$, but $(p_n, f(p_n)) \not\to (p, f(p)$.
$\therefore$ $(p, a) \neq (p, f(p))$, hence $a \notin f(E)$.
This implies that $(p, a) \notin Y$, contrary to compactness of Y.
$\therefore$ f is continuous on E.

Rudin Ch 4, p. 99 #7. If $E \subset X$ and if f is a function defined on X, the restriction of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$.
Define f and g on $\mathbb{R}^2$ by: $f(0, 0) = g(0, 0) = 0$, $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, $g(x, y) = \dfrac{xy^2}{x^2 + y^6}$ if $(x, y) \neq (0, 0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of (0, 0) and that f is not continuous at (0, 0); nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

Proof:
f is bounded on $\mathbb{R}^2$:

$0 \leq (x - y^2)^2 \Rightarrow 0 \leq x^2 - 2xy^2 + y^4 \Rightarrow 2xy^2 \leq x^2 + y^4 \Rightarrow \dfrac{xy^2}{x^2 + y^4} \leq 1/2.$

$\therefore f(x, y) \leq 1/2 \ \forall\ (x, y) \in \mathbb{R}^2.$

g is unbounded in every neighborhood of (0, 0):
Let $0 < \varepsilon < 1$. Let $|y| < \varepsilon$. Then $(|y^3|, |y|) \in B((0, 0), \varepsilon)$.
$g(|y^3|, |y|) = \left|\dfrac{y^3 y^2}{y^6 + y^6}\right| = \left|\dfrac{y^5}{2y^6}\right| = \left|\dfrac{1}{2y}\right|$. As $y \to 0$, $(y^3, y) \to (0, 0)$ and $g(|y^3|, |y|) = \left|\dfrac{1}{2y}\right| \to +\infty$.
$\therefore$ g is unbounded for $(x, y) \in B((0, 0), \varepsilon)$.

f is not continuous at (0, 0):
$\forall\ y \in \mathbb{R}, f(y^2, y) = 1/2$, but $f(0, y) \to 0$ as $y \to 0$.
$\therefore \lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$ does not exist.

Nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!
Let $m, b \in \mathbb{R}$.
$f(x, mx + b) = \dfrac{x(mx+b)^2}{x^2 + (mx+b)^4}$ and $g(x, mx + b) = \dfrac{x(mx+b)^2}{x^2 + (mx+b)^6}$.
For $b = 0$ we have $f(x, mx) \to 0$ and $g(x, mx) \to 0$ as $x \to 0$.
For $m = 0$ we have $f(x, b) \to 0$ and $g(x, b) \to 0$ as $x \to 0$.
For $b \neq 0$ and $m \neq 0$, the denominators are never 0, and so
$f|_{\{(x, mx+b)\}}$ and $g|_{\{(x, mx+b)\}}$ are continuous.

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Rudin, Ch 4 p. 99, #8. Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$.
Prove that f is bounded on E.
Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Proof:
$f : E \to \mathbb{R}, E \subset \mathbb{R}$.
Since f is uniformly continuous, then
$\exists\ \delta > 0$ such that $|f(p) - f(q)| < 1$, for all $p, q \in E$, for which $|p - q| < \delta$.
Since E is bounded, then $E \subset [a, b]$ for some $a, b \in \mathbb{R}$. We know $[a, b]$ is compact since it is closed and bounded and can hence be covered by a finite open cover.
We will construct this finite cover by letting $U_1 = [a, a + \delta)$,
$U_2 = \left(a + \frac{\delta}{2}, a + \frac{3\delta}{2}\right)$, $U_3 = (a + \delta, a + 2\delta)$, ..., $U_i = \left(a + \frac{(i-1)\delta}{2}, a + \frac{(i+1)\delta}{2}\right)$.
Since finitely many such open sets cover $[a, b]$, then finitely many such sets cover E.
Let n_0 be the largest index such that $U_{n_0} \cap [a, b] \neq \emptyset$.
Choose x, $y \in E$. Then for some $i, j \in \{1, 2, ..., n_0\}$, $x \in U_i$ and $y \in U_j$,
and $|x - y| \leq |j + 1 - i| \bullet \delta$.
So then by uniform continuity of f, $|f(x) - f(y)| < n_0$.
$\therefore\ f(E)$ is bounded.

Example:
Let $f : R \to R$ be defined by $f(x) = x$. Then f is uniformly continuous, but not bounded.

Rudin, Ch 4, p. 99 #10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n)), f(q_n)) > \varepsilon$. Use Theorem 2.37 (If E is an infinite subset of a compact set K, then E has a limit point in K.) to obtain a contradiction.

> Theorem 4.19: Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

Proof:
Let X, Y be metric spaces, X compact, and let $f: X \to Y$ be continuous.
Suppose f is not uniformly continuous and let $\delta > 0$.
Then $\exists\ \varepsilon > 0$ and $\exists\ p, q \in X$ such that $d_X(p, q) < \delta$ and $d_Y(f(p), f(q)) \geq \varepsilon$.
So then if we let $\delta_n = \dfrac{1}{n}$ for each $n \in \mathbb{N}$, then we can construct sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) < \delta_n$ but $d_Y(f(p_n), f(q_n)) \geq \varepsilon$.
Since $\{p_n\}$, $\{q_n\}$ are infinite subsets of a compact set X, then each has a limit point in X, say p_0 and q_0, respectively. And, $d_X(p_n, q_n) \to 0$, hence $p_0 = q_0$.
Since f is a continuous mapping of a compact metric space X into a metric space Y, then $f(X)$ is compact (Theorem 4.14).
$\therefore$ $f(p_n), f(q_n)$ are infinite subsets of a compact set $f(X)$; hence $f(p_n), f(q_n)$ have limit points in $f(X)$.
$\therefore$ $f(p_n) \to f(p_0)$ and $f(q_n) \to f(p_0)$, since f is continuous.
$\therefore$ $\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$, $d_Y(f(p_n), f(p_0)) < \varepsilon/2$ and $d_Y(f(q_n), f(p_0)) < \varepsilon/2$.
$\forall\ n \geq N$, $d_Y(f(p_n), f(q_n)) \leq d_Y(f(p_n), f(p_0)) + d_Y(f(q_n), f(p_0)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, contrary to our assumption that $d_Y(f(p_n), f(q_n)) \geq \varepsilon$.
$\therefore$ f is uniformly continuous on X.

Rudin, Ch 4 p. 99 #11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in exercise 13.

Proof:
Let $\{x_n\}$ be a Cauchy sequence in X and let $\varepsilon > 0$.
Since f is uniformly continuous, then
$\exists\ \delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.
Since $\{x_n\}$ is a Cauchy sequence, then $\exists\ N$ such that $\forall\ n, m \geq N$, $d_X(x_m, x_n) < \delta$.
It follows that $d_Y(f(x_m), f(x_n)) < \varepsilon$, $\forall\ n, m \geq N$.
$\therefore$ $\{f(x_n)\}$ is a Cauchy sequence in Y.

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Rudin, Ch 4, p. 100 #14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proof:
If $f(0) = 0$, we're done.
If $f(1) = 1$, we're done.
Then $f(0) > 0$ and $f(1) < 1$.
Let $g(x) = f(x) - x$.
This is continuous since it is a difference of 2 continuous functions.
So then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$.
$\therefore$ By the intermediate value theorem, $\exists\, x$ such that $g(x) = 0 = f(x) - x$.
$\therefore f(x) = x$.

Rudin, Ch 4, p. 101 #20. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by $\rho_E(x) = \inf\limits_{z \in E} d(x,z)$.
(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
(b) Prove that ρ_E is a uniformly continuous function on X, by showing that $|\rho_E(x) - \rho_E(y)| \le d(x, y)$ for all $x \in X, y \in X$.
Hint: $\rho_E(x) \le d(x, z) \le d(x, y) + d(y, z)$, so that $\rho_E(x) \le d(x, y) + \rho_E(y)$.

Proof:
(a) $\Rightarrow$:
$\rho_E(x) = \inf \{d(x, z) \mid z \in E\}$.
$\rho_E(x) = 0$ implies that $\inf \{ d(x, z) \mid z \in E\} = 0$.
$\therefore\ \forall\, \varepsilon > 0, \exists\, d(x, z) \in \rho_E(x)$ such that $d(x, z) < \varepsilon$.
$\therefore\ \forall\, \varepsilon > 0, \exists\, z \in E$ such that $d(x, z) < \varepsilon$.
$\therefore x \in E$ or $x \in E'$.
$\therefore x \in \overline{E}$.

$\Leftarrow$:
Assume $x \in \overline{E}$, then $x \in E$ or $x \in E'$.
If $x \in E$, then we're done.
Suppose $x \in E'$.
Then $\forall\, \varepsilon > 0, \exists\, z \in E$ such that $x \ne z$ and $d(x, z) < \varepsilon$.
$\therefore\ \forall\, \varepsilon > 0, \exists\, d(x, z) \in \rho_E(x)$ such that $d(x, z) < \varepsilon$.
$\therefore \inf \{d(x, z) \mid z \in E\} = 0$, which implies that $\rho_E(x) = 0$.

(b) Let $\varepsilon > 0$, let $\delta = \varepsilon$, let $x, y \in X$, and let $z \in E$.
Since $d(x, z) \le d(x, y) + d(y, z)$, then
$\rho_F(x) = \inf\{d(x, z) \mid z \in E\} \le \inf\{d(x, y) + d(y, z) \mid z \in E\} = d(x, y) + \rho_F(y)$. Also
$\rho_F(y) = \inf\{d(y, z) \mid z \in E\} \le \inf\{d(x, y) + d(x, z) \mid z \in E\} = d(x, y) + \rho_F(x)$.
Thus for $d(x, y) < \delta$ we have $|\rho_F(x) - \rho_F(y)| \le d(x, y) < \varepsilon$.
$\therefore\ \rho_E$ is a uniformly continuous function on X.

Week 2, Practice Problems, Assigned Friday, February 4, 2011

Rudin, Ch 4, p. 101 #21. Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$.
Hint: ρ_F is a continuous positive function on K.
Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Proof:
Let $p \in K, q \in F$.
By Exercise 20 (a), $\rho_F(p) = 0$ if and only if $p \in \overline{F} = F$. Since $K \cap F = \varnothing$, then $\rho_F(p) \neq 0$.
By Exercise 20 (b), ρ_F is continuous positive function on K.
Since K is compact, then $\rho_F(K)$ is also compact by Theorem 4.14.
Since $\rho_F(x) \neq 0$ for each $x \in K$, then compactness of K and continuity of ρ_F give us that $\min(\rho_F(K))$ exists, call it α. Let $\delta = \alpha/2$. Then $d(p, q) > \delta$.
$\therefore$ $\rho_F(p) = \delta$ for some $\delta > 0$, hence $d(p, q) \geq \delta$.

To show that the conclusion may fail for two disjoint closed sets if neither is compact, consider $K = \mathbb{Z}$ and $F = \{n + n/(n+1) \mid n \in \mathbb{N}\}$.
Then $\forall\ \delta > 0, \exists\ n \in \mathbb{N}$ such that $|n + 1 - (n + n/(n+1))| = 1/(n+1) < \delta$.