Week 5, Practice Problems, Assigned Friday, March 4, 2011

Chapter 5

1. Let $f \in C\,[0, 1]$. Assume f is differentiable on $(0, 1)$ and that $0 \le f(x) \le 1, \forall\, x \in [0, 1]$.

(a) Show that $\exists\, x_0 \in [0, 1] \ni f(x_0) = x_0$.

Proof:

Let $a = \min\limits_{0 \le x \le 1} f(x)$, let $b = \max\limits_{0 \le x \le 1} f(x)$.

If $f(a) = a$ or if $f(b) = b$, we're done.

Then $f(a) > a$ and $f(b) < b$.

Let $g(x) = f(x) - x$.

g is continuous since it is a difference of 2 continuous functions.

So then $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$.

$\therefore$ By the intermediate value theorem, $\exists\, t \in (0, 1)$ such that $g(t) = 0 = f(t) - t$.

$\therefore f(t) = t$.

(b) Show that x_0 is unique if $f'(x) \ne 1, \forall\, x \in (0, 1)$.

Proof:

Suppose there is more than one fixed point, so that $\exists\, x \ne y$ where $f(x) = x$ and $f(y) = y$.

Since f is differentiable, then $\exists\, c \in [0, 1]$ such that $f(x) - f(y) = f'(c)(x - y)$.

$\therefore$ $(x - y) = f'(c)(x - y)$.

$\therefore$ $f'(c) = 1$, which is a contradiction to our assumption that $f'(x) \ne 1, \forall\, x \in [0, 1]$.

$\therefore$ f has at most one fixed point.

QED

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2. Suppose that $f : (0, 1) \to \mathbb{R}$ is differentiable and bounded on $(0, 1)$.

Show that if $|f'(x)| < 1$ on $(0, 1)$, then $\lim\limits_{n \to \infty} f(\frac{1}{n})$ exists.

Proof:

Assume $|f'(x)| < 1$ on $(0, 1)$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N} \ni \forall\, n \ge N$, $\frac{1}{n} < \varepsilon$. Let $n \ge N$.

By the Mean Value Theorem, $\exists\, t \in (\frac{1}{n}, \frac{1}{N}) \ni \dfrac{f\left(\frac{1}{N}\right) - f\left(\frac{1}{n}\right)}{\frac{1}{N} - \frac{1}{n}} = f'(t)$.

Thus $\left|\dfrac{f\left(\frac{1}{N}\right) - f\left(\frac{1}{n}\right)}{\frac{1}{N} - \frac{1}{n}}\right| < 1$, hence $\left|f\left(\frac{1}{N}\right) - f\left(\frac{1}{n}\right)\right| < \left|\frac{1}{N} - \frac{1}{n}\right| < \frac{1}{N} < \varepsilon$.

Thus $\forall\, n \ge N$, $\left|f\left(\frac{1}{N}\right) - f\left(\frac{1}{n}\right)\right| < \varepsilon$. $\therefore$ $\lim\limits_{n \to \infty} f(\frac{1}{n})$ exists.

QED

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3. Show that if f is monotonic increasing on $[a, b]$ and it satisfies the intermediate value property, then f is continuous on $[a, b]$.

Proof:

Suppose $\exists\, p \in [a, b] \ni f$ is discontinuous. By corollary to Theorem 4.29, f cannot have discontinuities of the 2nd kind. And since f is monotonic increasing, it is not possible that $f(p-) = f(p+) \neq f(p)$. So we can only assume $f(p-) \neq f(p+)$.

Let $\varepsilon = f(p+) - f(p-)$. Choose $c \in (f(p+) - \varepsilon/4, f(p-) + \varepsilon/4) \ni c \neq f(p)$.
$\forall\, \delta, \forall\, x \in (p - \delta, p),\ f(x) \leq f(p-)$ and $\forall\, x \in (p, p + \delta),\ f(p-) \leq f(x)$.
So then $\nexists\, t \in (p - \delta, p + \delta) \ni f(t) = c$, contrary to the intermediate value property of f.

$\therefore\ f$ is continuous on $[a, b]$.

QED

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4. (a) Show that if f is differentiable on (a, b) and f' is bounded on (a, b), then f is uniformly continuous on (a, b).

Proof:

Let $x, y \in (a, b) \ni x < y$. By the Mean Value Theorem, $\exists\, t \in (x, y) \ni \dfrac{f(x) - f(y)}{x - y} = f'(t)$.

f' is bounded on $(a, b) \Rightarrow \exists\, M \in \mathbb{R}^+ \ni |f'(t)| \leq M$.

Thus $\left|\dfrac{f(x) - f(y)}{x - y}\right| \leq M$, hence $|f(x) - f(y)| \leq M\,|x - y|$.

Let $\varepsilon > 0$. Then for $\delta = \varepsilon/M$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq M\,|x - y| < M \bullet (\varepsilon/M) = \varepsilon$.
$\therefore\ f$ is uniformly continuous on (a, b).

(b) Give an example of a function which is differentiable and uniformly continuous on $(0, 1)$, but whose derivative is unbounded.

Example:

$f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is differentiable and uniformly continuous on $(0, 1)$ but has unbounded derivative.

Proof:

Since f is defined and continuous on $[0, 1]$, then $f([0, 1])$ is compact, hence uniformly continuous. However, $f'(x)$ is unbounded as x approaches 0.

QED

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5. Let $f:(0,+\infty) \to \mathbb{R}$. Suppose that f is differentiable on $(0,+\infty)$ and $f'(x) \geq c > 0$, $\forall\, x \in (0,+\infty)$. Prove that $\lim\limits_{x \to +\infty} f(x) = +\infty$.

Proof:

Let $x_0 \in (0,+\infty)$ and let $r \in \mathbb{R} \ni r > 1$.

By the Mean Value Theorem, $\exists\, t \in (x_0, x_0 + r) \ni \dfrac{f(x_0) - f(x_0 + r)}{x_0 - (x_0 + r)} = \dfrac{f(x_0 + r) - f(x_0)}{r} = f'(t)$.

By hypothesis, $\exists\, c \in \mathbb{R}^+ \ni c \leq f'(t)$.

Thus, $cr \leq f(x_0 + r) - f(x_0)$, hence $cr + f(x_0) \leq f(x_0 + r)$.

Let $M \in \mathbb{R}^+$. Then $\exists\, r \in \mathbb{R} \ni M \leq cr + f(x_0)$ and $M \leq f(x_0 + r)$.

$\therefore \lim\limits_{x \to +\infty} f(x) = +\infty$.

QED

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6. Suppose that f is differentiable on (a, b) and f' is increasing on (a, b). Prove that f' is continuous on (a, b).

Proof:

Let $x, y \in (a, b) \ni x < y$. Then f is differentiable on $[x, y]$.

By Theorem 5.12, f' has the intemediate value property for derivatives on $[x, y]$.

By Exercise 3 above, f' is continuous on (x, y).

Since x and y are arbitrary in (a, b), then f' is continuous on (a, b).

QED
