Real Analysis, Study Guide, Chapter 4

Limits of Functions

✳Rudin 4.1 **Definition**: $\lim_{x \to p} f(x) = q$ (Lecture notes 1/24/11)

Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or $\lim_{x \to p} f(x) = q$ if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$, $\exists\ \delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$.

It should be noted that $p \in X$ but that p need not be a point of E. Moreover, even if $p \in E$, we may very well have $f(p) \neq \lim_{x \to p} f(x)$.

✳Rudin 4.2 **Theorem** (Lecture notes 1/24/11)

Let X, Y, E, f, and p be as in Definition 4.1. Then $\lim_{x \to p} f(x) = q$ if an only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n \to \infty} p_n = p$.

Rudin **Corollary** to Theorem 4.2

If f has a limit at p, this limit is unique.

Rudin 4.3 **Definition**

$(f \pm g)(x) = f(x) \pm g(x)$; $(f \bullet g)(x) = f(x) \bullet g(x)$; $(f / g)(x) = f(x) / g(x)$, provided $g(x) \neq 0$; if $\lambda \in R$, then $(\lambda f)(x) = \lambda f(x)$.

✳Rudin 4.4 **Theorem** (Lecture notes 1/26/11)

Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and $\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$. Then

(a) $\lim_{x \to p} (f + g)(x) = A + B$

(b) $\lim_{x \to p} (f \bullet g)(x) = A \bullet B$

(c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

Continuous Functions

✳Rudin 4.5 **Definition**: *Continuous* (Lecture notes 1/26/11)

Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be continuous at p if for every $\varepsilon > 0$ there exists at $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E, then f is said to be *continuous* on E.

It should be noted that f has to be defined at the point p in order to be continuous at p.

✳Rudin 4.6 **Theorem**

In the situation given in Definition 4.5, assume also that p is a limit point of E. Then f is continuous at p is and only if $\lim_{x \to p} f(x) = f(p)$.

✳Rudin 4.7 **Theorem** (Lecture notes 1/26/11)

Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$ into Z, and h is the mapping of E into Z defined by $h(x) = g(f(x))$ $(x \in E)$.
If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.

✳Rudin 4.8 **Theorem** (Lecture notes 1/26/11)

A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Rudin **Corollary** to Theorem 4.8 (Lecture notes 1/26/11)

A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

✳Rudin 4.9 **Theorem** (Lecture notes 1/26/11)

Let f and g be complex continuous functions on a metric space X. The $f + g$, $f g$, and f/g are continuous on X.

Rudin 4.10 **Theorem** (Lecture notes 1/31/11)

(a) Let $f_1, \ldots, f_k$ be real functions on a metric space X, and let **f** be the mapping of X into R^k defined by $f(x) = (f_1(x), \ldots, f_k(x)($ $(x \in X)$; then **f** is continuous if and only if each of the functions $f_1, \ldots, f_k$ is continuous.
(b) If **f** and **g** are continuous mappings of X into R^k, then **f** + **g** and **f** • **g** are continuous on X.

Continuity and Compactness

Rudin 4.13 **Definition:** *bounded*

A mapping f of a set E into R^k is said to be *bounded* if there is a real number M such that $|f(x)| < M$ for all $x \in E$.

✳Rudin 4.14 **Theorem** (Lecture notes 1/31/11)

Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

✳Rudin 4.15 **Theorem** (Lecture notes 1/31/11)

If **f** is a continuous mapping of a compact space X into R^k, then **f**(X) is closed and bounded. Thus, **f** is bounded.

✳Rudin 4.16 **Theorem** (Lecture notes 1/31/11)

Suppose f is a continuous real function on a compact metric space X, and $M = \sup_{p\in x} f(p)$, $m = \inf_{p\in x} f(p)$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

✳Rudin 4.17 **Theorem** (Lecture notes 2/2/11)

Suppose f is a continuous 1–1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$, $(x \in X)$ is a continuous mapping of Y onto X.

✳Rudin 4.18 **Definition:** *Uniformly Continuous* (Lecture notes 2/2/11)

Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ for all p, q in X for which $d_x(p, q) < \varepsilon$.

✳Rudin 4.19 **Theorem** (Lecture notes 2/2/11)

Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

✳Rudin 4.20 **Theorem** (Lecture notes 2/2/11)

Let E be a noncompact set in R^1. Then

(a) There exists a continuous function on E which is not bounded;

(b) There exists a continuous and bounded function on E which has no maximum.

If, in addition, E is bounded, then

(c) There exists a continuous function on E which is not uniformly continuous.

✳Rudin 4.22 **Theorem** (Lecture notes 2/7/11)

If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

Corollary to Theorem 4.22

Let (X, d) be connected, $f: X \to R$ continuous. If $x, y \in X, f(x) < z < f(y)$, then $\exists\, c$ such that $f(c) = z$.

✳Rudin 4.23 **Theorem** (Intermediate Value Theorem) (Lecture notes 2/7/11)

Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

Discontinuities

✳Rudin 4.25 **Definition**: $f(x+), f(x-)$

Let f be defined on (a, b). Consider any point x such that $a \le x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequence $\{t_n\}$ in (x, b) such that $t_n \to x$. To obtain the definition of $f(x-)$, for $a < x \le b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x).

✳Rudin 4.26 **Definition**: *Discontinuity of the first/second kind*

Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a *discontinuity of the first kind*, or a simple discontinuity, at x. Otherwise the *discontinuity* is said to be *of the second kind*.

Definition *Discontinuity* (Lecture notes only 12/10/08)

We say c is a *discontinuity* of f if

(1) Either $f(c+)$ or $f(c-)$ does not exist

(2) $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist, but $\lim_{x \to c+} f(x) \neq \lim_{x \to c-} f(x)$

(3) $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) \neq f(c)$ (in which case c is a removable discontinuity)

Monotonic Functions

✳Rudin 4.28 **Definition**: *Monotonically increasing/decreasing*

Let f be real on (a, b). Then f is said to be *monotonically increasing* on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, we obtain the definition of a *monotonically decreasing* function. The class of monotonic functions consists of both the increasing and the decreasing functions.

✳Rudin 4.29 **Theorem** (Lecture notes 2/7/11)

Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point x of (a, b). More precisely, $\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$.

Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$.

Analogous results evidently hold for monotonically decreasing functions.

Rudin **Corollary** to Theorem 4.29

Monotonic functions have no discontinuities of the second kind.

✳Rudin 4.30 Theorem

Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.

✳Rudin 4.32 **Definition**: *Neighborhood of (+/–∞)*

For any real c, the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly $(-\infty, c)$ is a neighborhood of $-\infty$.

✳Rudin 4.33 **Definition**: *Convergence in the extended real number system*

Let f be a real function defined on $E \subset \mathbb{R}$. We say that $f(t) \to A$ as $t \to x$, where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of $x \ni V \cap E \neq \emptyset$, and such that $f(t) \in U \ \forall \ t \in V \cap E, t \neq x$.