Real Analysis, Study Guide, Chapter 5

The Derivative of a Real Function

✳Rudin 5.1 **Definition**: *Derivative/Differentiable* (Lecture Notes 2/9/11)

Let f be defined (and real valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient $\phi(t) = \dfrac{f(t) - f(x)}{t - x}$ $(a < t < b, t \neq x)$, and define $f'(x) = \lim_{t \to x} \phi(t)$, provided this limit exists.

We thus associate with the function f a function f' whose domain is the set of points x at which $\lim_{t \to x} \phi(t)$ exists; f' is called the *derivative* of f.

If f' is defined at a point x, we say that f is *differentiable* at x. If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

✳Rudin 5.2 **Theorem** (Lecture Notes 2/9/11)

Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.

✳✳Rudin 5.3 **Theorem** (Lecture Notes 2/9/11)

Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g, fg$, and f/g are differentiable on x, and

(a) $(f+g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f'(x)\, g(x) + f(x)g'(x)$;

(c) $(f/g)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$, $g(x) \neq 0$.

✳Rudin 5.4 **Examples** (Lecture Notes 2/9/11)

If $f(x) = c$, then $f'(x) = 0$.

If $f(x) = x$, then $f'(x) = 1$.

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Every polynomial is differentiable.

Every rational function is differentiable, except at the points where the denominator is 0.

✳✳Rudin 5.5 **Theorem** (Lecture Notes 2/9/11)

Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contais the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$ $(a \leq t \leq b)$, then h is differentiable at x, and $h'(x) = g'(f(x))\, f'(x)$.

Rudin 5.6 **Examples**

If $f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$, then $f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \quad (x \neq 0)$. And $f'(0)$ does not exist.

If $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$, then $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad (x \neq 0)$. And $f'(0) = 0$.

Thus, f is differentiable at all points x, but f' is not a continuous function, since $\cos(1/x)$ does not tend to a limit as $x \to 0$.

Mean Value Theorems

✳Rudin 5.7 **Definition**: *Local Maximum/Minima* (Lecture Notes 2/14/11)

Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Local minima are defined likewise.

✳✳Rudin 5.8 **Theorem** (Lecture Notes 2/9/11)

Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

✳✳Rudin 5.9 **Theorem** (Lecture Notes 2/14/11)

If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which $[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x)$.

✳✳Rudin 5.10 **Theorem** (Lecture Notes 2/14/11)

If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which $f(b) - f(a) = (b - a)\, f'(x)$.

✳✳Rudin 5.11 **Theorem** (Lecture Notes 2/14/11)

Suppose f is differentiable in (a, b).

(a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

The Continuity of Derivatives

✳✳Rudin 5.12 **Theorem** (Lecture Notes 2/14/11)

Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

A similar result holds if $f'(a) > f'(b)$.

Rudin **Corollary** to Theorem 5.12

If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.

L'Hospital's Rule

✳✳Rudin 5.13 **Theorem** (Lecture Notes 2/16/11)

Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or if $g(x) \to +\infty$ as $x \to a$, then $\frac{f(x)}{g(x)} \to A$ as $x \to a$.

Derivatives of Higher Order

Rudin 5.14 **Definition**: n^{th} *Derivative*

If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the 2^{nd} derivative of f. Continuing in this manner, we obtain functions $f, f', f'', f^{(3)}, \ldots, f^{(n)}$, each of which is the derivative of the preceding one. $f^{(n)}$ is called the n^{th} *derivative*, or the derivative or order n, of f.

Taylor's Theorem

Rudin 5.15 **Theorem** (Lecture Notes 2/16/11)

Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$. Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

Differentiation of Vector-Valued Functions

Rudin 5.16 **Remarks**

If f_1 and f_2 are the real and imaginary parts of f, that is, if $f(t) = f_1(t) + i \cdot f_2(t)$ for $a \le t \le b$, where $f_1(t)$ and $f_2(t)$ are real, then we have $f'(x) = f_1'(x) + i \cdot f_2'(x)$; also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x.

If $\mathbf{f} : [a, b] \to R^k$, then $\phi(t) = \dfrac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x}$ $(a < t < b, t \ne x)$, which is a point in R^k for each t and $\mathbf{f}'(x) = \lim_{t \to x} \phi(t)$. $\therefore$ $\mathbf{f}'(x)$ is that point of R^k for which

$$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = 0.$$

If $f_1, \ldots, f_k$ are the components of f, then $f' = (f_1', \ldots, f_k')$, and f is differentiable at a point x if and only if each of the functions $f_1, \ldots, f_k$ is differentiable at x.

Rudin 5.17 **Example**

Define, for real x, $f(x) = e^{ix} = \cos x + i \sin x$. Then $f(2\pi) - f(0) = 1 - 1 = 0$, but $f'(x) = ie^{ix}$, so that $|f'(x)| = 1$ for all real x. Thus, Theorem 5.10 fails to hold.

Rudin 5.18 **Example**

On the segment (0, 1), define $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$. Then $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = 1$, but

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$ $\therefore$ L'Hospital's Rule fails in this case.

Rudin 5.19 **Theorem**

Suppose $\mathbf{f}$ is a continuous mapping of $[a, b]$ into R^k and $\mathbf{f}$ is differentiable in (a, b). Then there exists $x \in (a, b)$ such that $|\mathbf{f}(b) - \mathbf{f}(a)| \le (b - a)|\mathbf{f}'(x)|$.