Content:

Definition 4.1 Limit of a function: $f(x) \to q$ as $x \to p$

Theorem 4.2 $\lim_{x \to p} f(x) = q \Leftrightarrow \forall \{p_n\} \subset E$ such that $p_n \neq p$, $\forall n \in \mathbb{N}$ and $p_n \to p$, it follows that $f(p_n) \to q$.

Final Exam Many got this wrong: $\sum_{n=1}^{\infty} \sin\frac{1}{n}$. As $n \to \infty$, $\frac{1}{n} \to 0$.

Discussion No one got this one right: $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$. Use the ratio test.

$$\frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \to e.$$

To prepare for the comprehensive exam, check a standard calculus text for problems on sequences and series.

Chapter 4

Discussion **Functions**

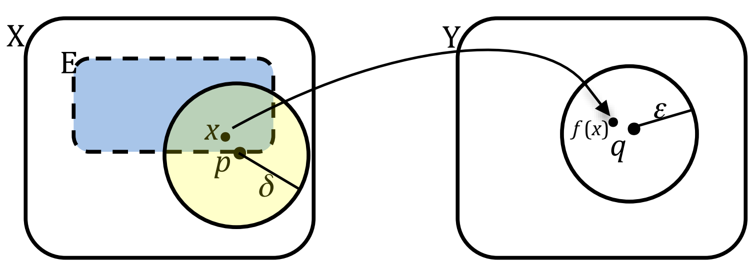
• To study continuity of functions, we study limits.
To study limits, we need the function to reside in a metric space.

• To study differentiability of functions, we look at $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
To study the derivative, we need the function to reside in a vector space.

$\lim_{h \to 0} \frac{f(x_0 + h \bullet v) - f(x_0)}{h}$ We look at what happens at the point x_0 when the vector, v, is multiplied by a very small number.

Definition Limit of a Function

Let X and Y be metric spaces. Let $E \subset X$, $f: E \to Y$, p is limit point of E.
Then $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q \Leftrightarrow$
$\forall\, \varepsilon > 0, \exists\, \delta = \delta_{(\varepsilon, p)} > 0$ such that $d(f(x), q) < \varepsilon\ \forall\, x \in E$ where $0 < d(x, p) < \delta$.



Example Let $E = (0, \infty)$ and let $Y = \mathbb{R}$. Define $f: E \to Y$ by $f(x) = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$.

Let $p = 0$. Then p is a limit point of E even though $p \notin E$.

And $\lim\limits_{x \to p} \dfrac{\sin\sqrt{x}}{\sqrt{x}} = 1$. So for $\varepsilon > 0$, $(1 - \varepsilon, 1 + \varepsilon)$ is the ε ball around $q = 1$, and $(0, \infty) \cap (-\delta, \delta) = (0, \delta)$ is the δ ball around $p = 0$.

Example Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} \dfrac{\sin x}{x} : x \neq 0 \\ 5 : x = 0 \end{cases}$.

Here $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, even though $f(x) = 5$.

Example Prove $\lim\limits_{x \to 3} x^2 = 9$. Choose an arbitrary $\varepsilon > 0$.

Solve $|x^2 - 9| < \varepsilon$ to find δ.

$-\varepsilon < x^2 - 9 < \varepsilon$

$-\varepsilon + 9 < x^2 < \varepsilon + 9$

$\sqrt{9 - \varepsilon} < x < \sqrt{9 + \varepsilon}$

We want $x - 3 < \delta$ to establish a ball around 3.

So $\sqrt{9 - \varepsilon} - 3 < x - 3 < \sqrt{9 + \varepsilon} - 3$.

If $\delta = \sqrt{9 + \varepsilon} - 3$, then $-\delta = 3 - \sqrt{9 + \varepsilon}$.

We can choose $\delta = \min\{\sqrt{9 + \varepsilon} - 3, 3 - \sqrt{9 - \varepsilon}\}$.

Then $|x - 3| < \delta$

$\Rightarrow |x - 3| < 3 - \sqrt{9 - \varepsilon}$ and $|x - 3| < \sqrt{9 + \varepsilon} - 3$

$\Rightarrow \sqrt{9 - \varepsilon} - 3 < x - 3 < \sqrt{9 + \varepsilon} - 3$

$\Rightarrow \sqrt{9 - \varepsilon} < x < \sqrt{9 + \varepsilon}$

$\Rightarrow -\varepsilon + 9 < x^2 < \varepsilon + 9$

$\Rightarrow -\varepsilon < x^2 - 9 < \varepsilon$

$\Rightarrow |x^2 - 9| < \varepsilon$.

A 2nd method:

$|x^2 - 9| < \varepsilon \Rightarrow |x - 3||x + 3| < \varepsilon \Rightarrow |x - 3| < \dfrac{\varepsilon}{|x + 3|}$.

We want x close to 3, so we only need to make sure $x + 3 \neq 0$ so that we can divide both sides by $|x + 3|$.

If $|x - 3| < 2$, then $-2 < x - 3 < 2$, and $4 < x + 3 < 8$.

So then $\dfrac{\varepsilon}{8} < \dfrac{\varepsilon}{|x + 3|}$. Choose $\delta = \min\{\dfrac{\varepsilon}{8}, 2\}$, then $|x - 3| < \delta$

$\Rightarrow |x - 3| < \dfrac{\varepsilon}{8} \Rightarrow |x^2 - 9| = |x - 3||x + 3| < \dfrac{\varepsilon}{8}|x + 3| < \dfrac{\varepsilon}{8} \bullet 8 = \varepsilon$.

Theorem 4.2 $\lim_{x \to p} f(x) = q \Leftrightarrow \forall \{p_n\} \subset E$ such that $p_n \neq p$, $\forall n \in \mathbb{N}$ and $p_n \to p$, it follows that $f(p_n) \to q$.

Proof:
$\Rightarrow$: Suppose $\lim_{x \to p} f(x) = q$. Then
$\forall \varepsilon > 0, \exists \delta = \delta_{(\varepsilon, p)} > 0$ such that $d(f(x), q) < \varepsilon \ \forall x \in E$ where $0 < d(x, p) < \delta$.
Let $\{p_n\} \subset E$ such that $p_n \neq p$, $\forall n \in \mathbb{N}$ and $p_n \to p$. Then
$\forall \delta > 0, \exists N_\delta \in \mathbb{N}$ such that $d(p_n, p) < \delta \ \forall n \geq N_\delta$.
Then put these two things together. Start with $\varepsilon > 0$.
Then choose δ according to the definition of limit of a function.
Choose $N_\varepsilon = N_{\delta(\varepsilon,\text{p})}$ according to the definition of limit of a sequence.
So $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ such that $d(f(p_n), q) < \varepsilon \ \forall n > N_\varepsilon$.
$\Leftarrow$: Will do on Wednesday.