Content:

Theorem 4.4 Let $E \subset X$, $Y = \mathbb{C}$, $p \in E'$; $f, g: E \to Y$, $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$. Then (a) $\lim_{x\to p}(f+g)(x) = A + B$, (b) $\lim_{x\to p}(fg)(x) = AB$, (c) $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

Definition Continuous function at p.

Theorem 4.7 Let X, Y, Z be metric spaces, $E \subset X$, $p \in E$, $f: E \to Y$, $g: f(E) \to Z$. If f is continuous at $p \in E$ and g is continuous at $f(p) \in f(E)$, then $g \circ f$ is continuous at $p \in E$.

Theorem 4.8 Let X, Y be metric spaces. Then $f: X \to Y$ is a continuous function $\Leftrightarrow f^{-1}(V)$ is open for every open set, $V \subset Y$.

Theorem 4.2 $\lim_{x\to p} f(x) = q \Leftrightarrow \forall \{p_n\} \subset E \ni p_n \to p, p_n \neq p, \forall n \in \mathbb{N}, f(p_n) \to q$.

Proof:

$\Leftarrow$: Suppose $\lim_{x\to p} f(x) \neq q$. Then

$\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0, \exists\, x_\delta \in E \ni 0 < d(x_\delta, p) < \delta$ but $d(f(x), q) \geq \varepsilon$.

Let $\delta = \frac{1}{n}$. Then $\exists\, x_n \in E \ni 0 < d(x_n, p) < \frac{1}{n}$ but $d(f(x), q) \geq \varepsilon$.

This is a sequence whose limit is p, but $f(p_n) \not\to q$.

$\therefore \Leftarrow$ holds true.

Note $\Rightarrow$: is always true.

$\Leftarrow$: is true only for metric spaces. Some general spaces are not enough.

Corollary $\lim_{x\to p} f(x) = q$ is unique if it exists.

Theorem **4.4** Consider $E \subset X$, $Y = \mathbb{C}$, $p \in E'$. Also let $f, g: E \to Y$. Suppose $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then

(a) $\lim_{x\to p}(f+g)(x) = A + B$,

(b) $\lim_{x\to p}(fg)(x) = AB$,

(c) $\lim_{x\to p}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$. (We don't need $g(x) \neq 0$ here.)

Notation $(f+g)(x) \overset{defn}{=} f(x) + g(x)$.

Definition Let X, Y be metric spaces, $E \subset X$, $p \in E$, $f: E \to Y$.
f is *continuous* at $p \in E \Leftrightarrow$
$\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\forall\, x \in E$ where $\underbrace{d(x, p) < \delta}$ we have $d\underbrace{(f(x), f(p))} < \varepsilon$.

$$x \in B(p, \delta) \cap E \Rightarrow f(x) \in B(f(p), \varepsilon)$$
$$f(B(p, \delta) \cap E) \subset B(f(p), \varepsilon)$$

Comments Case 1: $p \in E'$. f is continuous at p if $\lim_{x \to p} f(x) = f(p)$.

Case 2: $p \in (E \setminus E')$. $\exists\, r_0 \ni B(p, r_0) \cap E = \{p\}$. So $\forall\, \varepsilon > 0$, choose $\delta = r_0$.

Definition f is continuous on E if f is continuous at every point $p \in E$.

Theorem **4.7** Let X, Y, Z be metric spaces, $E \subset X$, $p \in E$, $f: E \to Y$, $g: f(E) \to Z$.
If f is continuous at $p \in E$ and g is continuous at $f(p) \in f(E)$, then
$g \circ f$ is continuous at $p \in E$.
Proof:
For $\varepsilon > 0$, $\exists\, \delta > 0 \ni g(B(f(p), \delta) \cap f(E)) \subset B(g(f(p)), \varepsilon)$.
For $\delta > 0$, $\exists\, \eta > 0 \ni f(B(p, \eta) \cap E) \subset B(f(p), \delta)$. Note that this is equivalent to $f(B(p, \eta) \cap E) \subset B(f(p), \delta) \cap f(E)$.
So then $g(f(B(p, \eta) \cap E)) \subset g(B(f(p), \delta) \cap f(E)) \subset B(g(f(p)), \varepsilon)$.
Thus $g \circ f(B(p, \eta) \cap E) \subset B(g \circ f(p), \varepsilon)$.

Theorem **4.8** Let X, Y be metric spaces. Then $f: X \to Y$ is a continuous function
$\Leftrightarrow f^{-1}(V)$ is open for every open set, $V \subset Y$.
Proof:
$\Rightarrow$: Suppose f is continuous. Let $V \subset Y$ be open.
Case 1: If $f^{-1}(V) = \varnothing$, then $f^{-1}(V)$ is open.
Case 2: If $f^{-1}(V) \neq \varnothing$, then $\exists\, p \in f^{-1}(V)$. Thus $f(p) \in V$.
V is open $\Rightarrow \exists\, \varepsilon > 0 \ni B(f(p), \varepsilon) \subset V$.
f is continuous $\Rightarrow \exists\, \delta > 0 \ni f(B(p, \delta)) \subset B(f(p), \varepsilon) \subset V$.
So then $B(p, \delta) \subset f^{-1}(f(B(p, \delta))) \subset f^{-1}(V)$, hence $f^{-1}(V)$ is open.
$\Leftarrow$: Let $p \in X$. Let $\varepsilon > 0$. Consider $V = B(f(p), \varepsilon)$.
Assume $f^{-1}(V)$ is open.
Since $p \in f^{-1}(B(f(p), \varepsilon))$, then p is an interior point.
Thus $\exists\, \delta > 0 \ni B(p, \delta) \subset f^{-1}(B(f(p), \varepsilon))$.
So then $f(B(p, \delta)) \subset f(f^{-1}(B(f(p), \varepsilon)) \subset B(f(p), \varepsilon)$.
$\therefore f$ is continuous.

Note In the proof we used the following: $f(f^{-1}(V)) \subset V$.
Proof:
$y \in f(f^{-1}(V)) \Rightarrow \exists\, x \in f^{-1}(V) \ni y = f(x) \Rightarrow f(x) \in V \Rightarrow y \in V$.