Content:

Theorem 4.10 (a) Let $f_1, f_2, ..., f_k : X \to \mathbb{R}$ be functions. Then $F: X \to \mathbb{R}^k$ defined by $F(x) = (f_1(x), f_2(x), ..., f_k(x))$ is continuous $\Leftrightarrow$ each of $f_1, f_2, ..., f_k$ are continuous.
(b) If $F, G : X \to \mathbb{R}^k$ are continuous, then $F + G$, $F \bullet G$ are continuous.
Note that $F + G$ is a mapping, $F \bullet G$ is real valued.

Definition 4.13 Bounded

Theorem 4.14 Let X be a compact metric space and Y a metric space.
If $f : X \to Y$ is continuous, then $f(X)$ is compact.

Theorem 4.16 Let X be a compact metric space and $f : X \to \mathbb{R}$ be continuous.
Then $\exists\, p, q \in X \ni f(p) = \min_{x\in X} f(x)$ and $f(q) = \max_{x\in X} f(x)$.

Homework #7, $f(x) = \left\{ \sum_{k=1}^{\infty} \frac{1}{2^k} \sqrt{|x - r_k|} \right.$ defined on (0, 1). This is continuous.

Discussion $f(x) = \left\{ \sum_{k=1}^{\infty} \frac{1}{2^k} \bullet \frac{1}{\sqrt{|x - r_k|}} \right.$ blows up if $x = r_k$. So there are infinitely many spikes on the interval (0, 1), yet the area is still finite.

Consider $\int_{k=1}^{\infty} \frac{1}{|x - 1/2|} dx$. This is an improper integral.

But $\int_{k=1}^{\infty} \frac{1}{\sqrt{x - 1/2}} dx$ is a convergent improper integral.

Theorem **4.10 (a)** Let $f_1, f_2, ..., f_k : X \to \mathbb{R}$ be functions.
Then $F: X \to \mathbb{R}^k$ defined by $F(x) = (f_1(x), f_2(x), ..., f_k(x))$ is continuous $\Leftrightarrow$ each of $f_1, f_2, ..., f_k$ are continuous.
(b) If $F, G : X \to \mathbb{R}^k$ are continuous, then $F + G$, $F \bullet G$ are continuous.
Note that $F + G$ is a mapping, $F \bullet G$ is real valued.

(a) Proof:
$\Rightarrow$: Assume F is continuous. Let $p \in X$. Let $\varepsilon > 0$.
Then $\exists\, \delta > 0 \ni |x - p| < \delta \Rightarrow ||F(x) - F(p)|| < \varepsilon$.
Since $|f_i(x) - f_i(p)| \le ||F(x) - F(p)||\ \forall\, i \in \{1, 2, ..., k\}$, then each of $f_1, f_2, ..., f_k$ are continuous.
$\Leftarrow$: Assume each of $f_1, f_2, ..., f_k$ are continuous at p.
Then $\exists\, \delta > 0 \ni |x - p| < \delta \Rightarrow |f_i(x) - f_i(p)| < \varepsilon/k$.
Since $||F(x) - F(p)|| \le \sum_{i=1}^{k} |f_i(x) - f_i(p)| < k \bullet \varepsilon/k$, then F is continuous.

Note We have applied Hölder's Inequality, $\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \bullet \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$
where $1/p + 1/q = 1$, to get
$||F(x) - F(p)|| = \left(\sum_{i=1}^{k} |f_i(x) - f_i(p)|^2\right)^{\frac{1}{2}} \le \sum_{i=1}^{k} |f_i(x) - f_i(p)|$.

Example **4.11** Read in text.

Definition **4.13** $f : E \to \mathbb{R}^k$, $E \subset X$. f is *bounded* on E if $\exists\, M \in \mathbb{R} \ni M \geq 0$ and $\|f(x)\| \leq M\ \forall\, x \in E$.

Note We can bound on sets that do not have a topological structure, so it is not necessary for X to be a metric space.

Theorem **4.14**
Let X be a compact metric space and Y a metric space.
If $f : X \to Y$ is continuous, then $f(X)$ is compact.
(that is, continuous functions carry compact sets into compact sets.)

Proof:
Consider an open cover of $f(X)$.
Then $f(X) \subset \bigcup_{\alpha \in A} V_\alpha$ where each V_α is open in Y.
$X \subset f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$.
Hence $\bigcup_{\alpha \in A} f^{-1}(V_\alpha)$ is an open cover of X. Since X is compact then we can choose a finite subcover such that $X \subset \bigcup_{i=1}^{n} f^{-1}(V_i)$.
So then $f(X) \subset f\left(\bigcup_{i=1}^{n} f^{-1}(V_i)\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(V_i)\right) \subset \bigcup_{i=1}^{n} V_i$,
a finite subcover of $f(X)$. $\therefore$ $f(X)$ is compact.

Remark Is $f^{-1}(K)$ compact if $K \subset Y$ is compact?
Yes, if X is compact. No, if X is not compact.

Theorem **4.15**
If $f : X \to \mathbb{R}^k$, X is compact, and f is continuous, then $f(X)$ is bounded and closed. And so f is bounded.

Theorem **4.16**
Let X be a compact metric space and $f : X \to \mathbb{R}$ be continuous.
Then $\exists\, p, q \in X \ni f(p) = \min_{x \in X} f(x)$ and $f(q) = \max_{x \in X} f(x)$.

Proof:
$f(x)$ is bounded and closed in $\mathbb{R}$. Let $m = \inf f(x) \in f(X)$.
Then $\exists\, p \in X \ni f(p) = m$.
Similarly $\exists\, q \in X \ni f(q) = M = \max_{x \in X} f(x)$.

Example $\sup(\tan^{-1}(x)) = \pi/2$, but $\tan^{-1}(x)$ has no maximum.