Content:

Theorem 4.22 Let X, Y be metric spaces and $E \subset X$ be a connected set. If $f: X \to Y$ is continuous, then $f(E)$ is connected.

Theorem 4.23 Let $f:[a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < f(b)$, and $f(a) < c < f(b)$, then $\exists\, x_c \in (a, b) \ni f(x_0) = c$.

Theorem 4.29 If $f: (a, b) \to \mathbb{R}$ is monotonically increasing, then $f(x+)$ and $f(x-)$ exist. Moreover, $\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) \le \inf_{x<t<b} f(t)$.

Example $f(x) = x^2$ is not uniformly continuous.

If it was, then $\forall\, \varepsilon > 0, \exists\, \delta(\varepsilon) > 0 \ni |x - y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$.

$\delta(\varepsilon) = \inf_{x>0} \delta(\varepsilon, x) > 0$. However,

$\exists\, \varepsilon > 0 \ni \forall\, \delta > 0\ \exists\, x_\delta, y_\delta > 0 \ni |x_\delta - y_\delta| < \delta$ but $|x_\delta^2 - y_\delta^2| \ge \varepsilon$.

Let $\varepsilon = 1$ and let $\delta = \frac{1}{n}$. Then for $x_n = n - \frac{1}{4n}$ and $y_n = n + \frac{1}{4n}$, we have

$|x_n^2 - y_n^2| = |x_n - y_n| \bullet |x_n + y_n| = \frac{1}{2n} \bullet 2n = 1 \ge \varepsilon$.

This is example can be used for Week 2 #5.

Note $f: [0, +\infty) \to \mathbb{R}, f(x) = x^2$ is not uniformly continuous.

$f: [0, M) \to \mathbb{R}, f(x) = x^2$ is uniformly continuous.

$f: \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{2, 5]$ is also uniformly continuous.

Discontinuities

Example **(1)** $f(x) = \begin{cases} 1 : x \in Q \\ 0 : x \notin Q \end{cases}$.

We can make irrational and rational sequences that converge to the same point, $x \in \mathbb{R}$, yet the function values of each sequence cannot both converge to 1 or 0.

(2) $f(x) = \begin{cases} x : x \in Q \\ 0 : x \notin Q \end{cases}$.

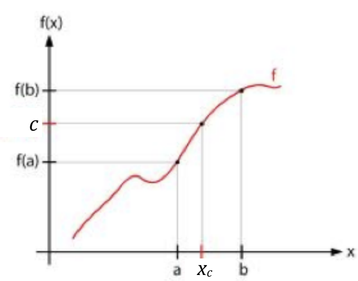
This is continuous at 0, as the function values of any irrational sequences stays constant at 0, and any rational sequence that converges to 0 has function values converging to 0.

Theorem **4.22** Let X, Y be metric spaces and $E \subset X$ be a connected set. If $f: X \to Y$ is continuous, then $f(E)$ is connected.

Proof:
Suppose $f(E)$ is disconnected.
Then $f(E) = A \cup B$ where $\overline{A} \cap B = \varnothing = A \cap \overline{B}$, and $A, B \neq \varnothing$.
$E = f^{-1}(f(E)) \cap E = f^{-1}(A \cup B) \cap E = (f^{-1}(A) \cap E) \cup (f^{-1}(B) \cap E)$.
$A, B \neq \varnothing \Rightarrow (f^{-1}(A) \cap E) \neq \varnothing$ and $(f^{-1}(B) \cap E) \neq \varnothing$ and
$\overline{A} \cap B = \varnothing \Rightarrow$
$\overline{(f^{-1}(A) \cap E)} \cap (f^{-1}(B) \cap E)$
$\subset (\overline{f^{-1}(A)} \cap \overline{E}) \cap (f^{-1}(B) \cap E)$
$= \overline{f^{-1}(A)} \cap f^{-1}(B) \cap \overline{E} \cap E$
$= \overline{f^{-1}(A)} \cap f^{-1}(B) \cap E$
$\subset \overline{f^{-1}(\overline{A})} \cap f^{-1}(B) \cap E$
$= f^{-1}(\overline{A}) \cap f^{-1}(B) \cap E$ (by continuity of f)
$= f^{-1}(\overline{A} \cap B) \cap E$
$= f^{-1}(\varnothing) \cap E$
$= \varnothing$.
By similar proof we have $(f^{-1}(\overline{A}) \cap E) \cap \overline{(f^{-1}(B) \cap E)} = \varnothing$.
$\therefore$ $(f^{-1}(A) \cap E) \cup (f^{-1}(B) \cap E)$ forms a separation of E, contrary to our assumption that E is connected.

Theorem **4.23** Let $f: [a, b] \to \mathbb{R}$ be a continuous function.
If $f(a) < f(b)$, and $f(a) < c < f(b)$, then
$\exists\, x_c \in (a, b) \ni f(x_c) = c$.

Proof:
Think about it.



Example To show the reverse (having the ivp $\Rightarrow f$ is continuous) is not true,

look at $f(x) = \begin{cases} \sin\frac{1}{x} : 0 < x < 1 \\ 1 : x = 1 \end{cases}$.

Theorem **4.29** If $f : (a, b) \to \mathbb{R}$ is monotonically increasing, then $f(x+)$ and $f(x-)$ exist. Moreover, $\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) \le \inf_{x<t<b} f(t)$.

Proof:
Let $x \in (a, b)$.
Let $A = \{f(t) \mid a < t < x\}$.
A is bounded above by $f(b)$ so $\alpha = \sup A$ exists.
By definition of supremum, $\alpha \le f(x)$.
So we only need to show $\alpha = f(x-)$.
Let $\varepsilon > 0$.
Then $\exists\, z_\varepsilon \in A \ni \alpha - \varepsilon < z_\varepsilon \le \alpha$. And $z_\varepsilon = f(x - \delta(\varepsilon))$ where $a < x - \delta(\varepsilon) < x$.
Then $\alpha - \varepsilon < f(x - \delta(\varepsilon)) \le \alpha$.
This and the monotonicity of f gives us that
$a < x - \delta(\varepsilon) < t < x \Rightarrow \alpha - \varepsilon < f(x - \delta(\varepsilon)) \le f(t) \le \alpha \Rightarrow |\alpha - f(t)| < \varepsilon$.
$\therefore \sup_{a<t<x} f(t) = f(x-)$.
By similar proof, $f(x+) \le \inf_{x<t<b} f(t)$.