Content:

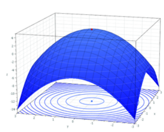
Definition	Local maximum
Theorem	5.8 For $f: [a, b] \to \mathbb{R}$, if f has a local max(min) at $x_0 \in (a, b)$ and $f'(x_0)$ exists, then $f'(x_0) = 0$.
Corollary	(Rolle's Theorem) If $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then $\exists\, x_0 \in (a, b) \ni f'(x_0) = 0$.
Theorem	5.9 (Cauchy's Theorem) If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then $\exists\, x_0 \in (a, b) \ni$ $[f(b) - f(a)]g'(x_0) = [g(b) - g(a)]f'(x_0)$.
Corollary	5.10 (Lagrange's Theorem) If $f:[a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable in (a, b), then $\exists\, x_0 \in (a, b) \ni f(b) - f(a) = f'(x_0)(b - a)$.
Theorem	5.11 For f is differentiable on (a, b), (a) If $f'(x) > 0\ \forall\, x \in (a, b)$, f is strictly increasing; (b) If $f'(x) = 0\ \forall\, x \in (a, b)$, f is constant; and (c) If $f'(x) < 0\ \forall\, x \in (a, b)$, f is strictly decreasing.
Theorem	5.12 (IVP of $f'(x)$) If f is differentiable on $[a, b]$ and $\lambda \in \mathbb{R} \ni$ $f'(a) < \lambda < f'(b)$, then $\exists\, x_0 \in (a, b) \ni f'(x_0) = \lambda$.
Corollary	If f is differentiable on $[a, b]$, then f' cannot have discontinuities of the 1st kind on $[a, b]$.

Homework Correction Week #3, #1, Change $B(x, r)$ to $B(x_0, r)$.

Mean Value Theorems

Definition **5.7** Suppose $f: X \to \mathbb{R}$, X a metric space. We say $x_0 \in X$ is a *local maximum point* if $\exists\, r > 0 \ni f(x) \le f(x_0)\ \forall\, x \in B(x_0, r)$.

Example $f: \mathbb{R}^2 \to \mathbb{R}$.



Theorem **5.8** Let $f: [a, b] \to \mathbb{R}$ be given. If f has a local maximum(minimum) at $x_0 \in (a, b)$ and $f'(x_0)$ exists. Then $f'(x_0) = 0$.

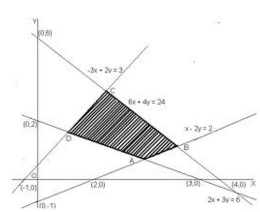
Proof:

$$0 \le \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(x_0 - h) - f(x_0)}{-h} = f'(x_0) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(x_0 + h) - f(x_0)}{h} \le 0.$$

$\therefore\ f'(x_0) = 0$.

Remark $a < x_0 < b$ is essential as, in the case that $f(b)$ is the local maximum, $f'(b)$ is not necessarily 0.
And $f'(x_0)$ exists is essential.

Example Variational Inequalities
$a < x_0 < b$ is not essential, but
$f(b)(x - a) \ge 0\ \forall\, x \in [a, b]$ is.



Remark $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, \ldots, x_n) =$

$$\frac{\partial f}{\partial x_2}(x_{0_1}, x_{0_2}, \ldots, x_{0_n}) = \lim_{h \to 0} \frac{f(x_{0_1}, x_{0_2} + h, \ldots, x_{0_n}) - f(x_{0_1}, x_{0_2}, \ldots, x_{0_n})}{h}.$$

If f has a local max(min) at $(x_{0_1}, x_{0_2}, \ldots, x_{0_n}) \in \text{interior}(D)$, then

$$\frac{\partial f}{\partial x_1}(x_{0_1}, x_{0_2}, \ldots, x_{0_n}) = \frac{\partial f}{\partial x_n}(x_{0_1}, x_{0_2}, \ldots, x_{0_n}) = 0.$$

Corollary **Rolle's Theorem**

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) and $f(a) = f(b)$, then $\exists\, x_0 \in (a, b) \ni f'(x_0) = 0$.

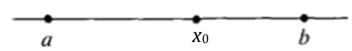
Proof:

f is continuous on $[a, b]$. So then $\exists\, x_0 \in [a, b] \ni f(x_0) = \max\limits_{a \le x \le b} f(x)$.

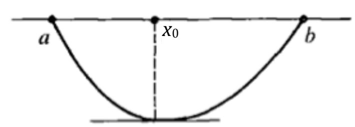
Case 1: $x_0 = a$ or $x_0 = b$.

Then $\exists\, x_1 \in [a, b] \ni f(x_1) = \min\limits_{a \le x \le b} f(x)$.

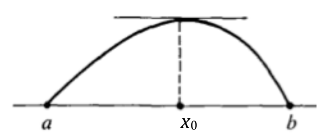
(a) If $x_1 = a$ or $x_1 = b$, then $f(x) = c\ \forall\, x \in [a, b]$, for some $c \in \mathbb{R}$.



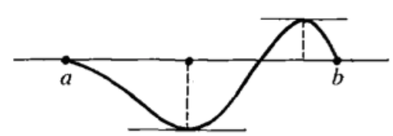
(b) If $a < x_1 < b$ then $f'(x_1) = 0$.



Case 2: $a < x_0 < b$. Then $f'(x_0) = 0$.



Note that there may be more than one max, min, or both:



Theorem **5.9** Cauchy's Theorem

If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then $\exists\, x_0 \in (a, b) \ni [f(b) - f(a)]g'(x_0) = [g(b) - g(a)]f'(x_0)$.

Proof:

Define $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$.

Then h is continuous on $[a, b]$ and differentiable in (a, b).

Note that $h(a) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a)$

$= f(b)g(a) - g(b)f(a) = f(b)g(b) - g(b)f(a) - g(b)f(b) - f(b)g(a)$

$= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) = h(b)$.

By Rolle's Theorem, $\exists\, x_0 \in (a, b) \ni$

$h'(x_0) = [f(b) - f(a)]g'(x_0) - [g(b) - g(a)]f'(x_0) = 0$.

$\therefore\ [f(b) - f(a)]g'(x_0) = [g(b) - g(a)]f'(x_0)$.

Corollary **5.10** (Lagrange's Theorem)

If $f:[a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable in (a, b), then $\exists\, x_0 \in (a, b) \ni f(b) - f(a) = f'(x_0)(b - a)$.

Proof:

Let $g(x) = x$. Then g is continuous on $[a, b]$ and differentiable in (a, b).
By Cauchy's Theorem, $\exists\, x_0 \in (a, b) \ni$
$f(b) - f(a) = [f(b) - f(a)]g'(x_0) = [g(b) - g(a)]f'(x_0) = f'(x_0)(b - a)$.

Theorem **5.11** Suppose f is differentiable on (a, b), then

(a) If $f'(x) > 0\ \forall\, x \in (a, b)$, f is strictly increasing.
(b) If $f'(x) = 0\ \forall\, x \in (a, b)$, f is constant.
(c) If $f'(x) < 0\ \forall\, x \in (a, b)$, f is strictly decreasing.

Proof:

Consider $a < x_1 < x_2 < b$.
By Lagrange's Theorem, $f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1)$ for some $x_0 \in (x_1, x_2)$.
(a) $f'(x_0) > 0$ and $(x_2 - x_1) > 0 \Rightarrow f(x_2) - f(x_1) > 0$, hence $f(x_2) > f(x_1)$.
(b) $f'(x_0) = 0$ and $(x_2 - x_1) > 0 \Rightarrow f(x_2) - f(x_1) = 0$, hence $f(x_2) = f(x_1)$.
(c) $f'(x_0) < 0$ and $(x_2 - x_1) > 0 \Rightarrow f(x_2) - f(x_1) < 0$, hence $f(x_2) < f(x_1)$.

Theorem **5.12** (Intermediate Value Theorem for Derivatives)

Suppose f is differentiable on $[a, b]$ and let $\lambda \in \mathbb{R} \ni f'(a) < \lambda < f'(b)$.
Then $\exists\, x_0 \in (a, b) \ni f'(x_0) = \lambda$.

Proof:

Let $g(t) = f(t) - \lambda t$. Then $g'(t) = f'(t) - \lambda$.
Since $f'(a) < \lambda$, then $g'(a) = f'(a) - \lambda < 0$.
And since $f'(b) > \lambda$, then $g'(b) = f'(b) - \lambda > 0$.
By Theorem 5.11 (a) and (c) we have
$\exists\, x_1 > a \ni g(x_1) < g(a)$ and $\exists\, x_2 < b \ni g(x_2) < g(b)$.
By continuity of f on $[a, b]$ and Theorem 4.16, $\exists\, x_0 \in [a, b] \ni g(x_0) = \min_{0 \le x \le b} g(x)$.
Note that $g(x_1) < g(a)$ for $a < x_1$ and $g(x_2) < g(b)$ for $x_2 < b \Rightarrow$
$x_0 \neq a$ and $x_0 \neq b$. Thus $x_0 \in (a, b)$.
Then by Theorem 5.8 above, $g'(x_0) = 0$, hence $f'(x_0) = \lambda$.

Corollary If f is differentiable on $[a, b]$, then f' cannot have discontinuities of the 1st kind on $[a, b]$.

Proof:
If f' has a discontinuity of the 1st kind, then
$\exists\, x \in (a, b) \ni f'(x+)$ and $f'(x-)$ exist but $f'(x+) \neq f'(x-)$ or
$f'(x) \neq f'(x+) = f'(x-)$.
Suppose $f'(x+) \neq f'(x-)$ and let $\varepsilon = |f'(x-) - f'(x+)|$.

By definition of $f'(x-)$ and $f'(x+)$,
$\exists\, \delta_1 > 0 \ni x - \delta_1 < t < x \Rightarrow |f'(t) - f'(x-)| < \varepsilon/4$ and
$\exists\, \delta_2 > 0 \ni x < t < x + \delta_2 \Rightarrow |f'(t) - f'(x+)| < \varepsilon/4$.

Choose $c \in (f'(x-) + \varepsilon/4, f'(x+) - \varepsilon/4)$.

Then $\forall\, t \in (x - \delta_1, x + \delta_2)$,

$f'(t) \in (f'(x-), f'(x-) + \varepsilon/4) \cup (f'(x+) - \varepsilon/4, f'(x+))$.
$\therefore\ f'(t) \neq c$, hence f' does not have the IVP.

Suppose $f(x) \neq f(x+) = f(x-)$. Let $\varepsilon = |f'(x) - f'(x+)|$.
Then $\exists\, \delta > 0 \ni x - \delta < t < x \Rightarrow |f'(t) - f'(x+)| < \varepsilon/4$.

Let $m = \min\{f'(x), f'(x+)\}$, and $M = \max\{f'(x), f'(x+)\}$.
Choose $c \in (m + \varepsilon/4, M - \varepsilon/4)$.
Then $\forall\, t \in (x - \delta, x), f(t) \in (m, m + \varepsilon/4) \cup (M - \varepsilon/4, M)$.
$\therefore\ f'(t) \neq c$, hence f' does not have the IVP.

Discussion The Heaviside function is defined as follows:

$H(x) = \begin{cases} 0 : x < 0 \\ 1 : x \geq 0 \end{cases}$. This function has applications in engineering.

The Dirac delta function is defined as follows:

$\delta(x) = \begin{cases} 0 : x \neq 0 \\ \infty : x = 0 \end{cases}$. This function has no meaning mathematically

however, it works very well in engineering applications.
Form the sequence of areas such that the area under the graph is always 1, but the domain decreases to 0 while the range increases to ∞.