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Theorem 5.15 Taylor's Theorem (continued)

Theorem 5.19 Let $\mathbf{f}$: $[a, b] \to \mathbb{R}^k$ be a vector valued continuous function (curve in $\mathbb{R}^k$). If $\mathbf{f}$ is differntiable in (a, b), then $\exists\, x \in (a, b) \ni |\mathbf{f}(b) - \mathbf{f}(a)| \le (b - a)|\mathbf{f}'(x)|$.

Theorem **5.15 Taylor's Theorem**

Let f: $[a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ continuous on $[a,b]$ and $f^{(n)}(t)$ exists $\forall\, t \in (a,b)$, then $\forall\, \alpha, \beta \in [a,b]$ $\exists\, \alpha < x < \beta \ni f(\beta) = P(\beta) + \dfrac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$ where $P(t) = \sum_{k=0}^{n-1} \dfrac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$.

Proof:

For $n = 1$, $P(t) = \dfrac{f^{(0)}(\alpha)}{0!}(t-\alpha)^0 = f(\alpha)$ and Cauchy's Mean Value Theorem gives us that $\exists\, x \in (\alpha, \beta) \ni \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(x)$. And $f(\beta) = f(\alpha) + f'(x)(\beta - \alpha) = P(\beta) + (f'(x)/1!)(\beta - \alpha)^1$ and the theorem holds.

Let $n > 1$ and define $M \in \mathbb{R}$ by $f(\beta) - P(\beta) = M(\beta - \alpha)^n$.

Define $g(t) = f(t) - P(t) - M(t - \alpha)^n$.

Note that $P(t) = f(\alpha) + f'(\alpha)(t-\alpha) + \dfrac{f^{(2)}(\alpha)}{2!}(t-\alpha)^2 + \cdots + \dfrac{f^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}$,

$P'(t) = f'(\alpha) + f^{(2)}(\alpha)(t-\alpha) + \dfrac{f^{(3)}(\alpha)}{2!}(t-\alpha)^2 + \cdots + \dfrac{f^{(n-1)}(\alpha)}{(n-2)!}(t-\alpha)^{n-2}$,

$P^{(2)}(t) = f^{(2)}(\alpha) + \dfrac{f^{(3)}(\alpha)}{1!}(t-\alpha) + \dfrac{f^{(4)}(\alpha)}{2!}(t-\alpha)^2 + \cdots + \dfrac{f^{(n-1)}(\alpha)}{(n-3)!}(t-\alpha)^{n-3}$,

$P^{(n-1)}(t) = f^{(n-1)}(\alpha)$. And so $P^{(n)}(t) = 0$.

$g'(t) = f'(t) - P'(t) - nM(t - \alpha)^{n-1}$,

$g''(t) = f''(t) - P''(t) - n(n-1)M(t - \alpha)^{n-1}$, ...,

$g^{(n)}(t) = f^{(n)}(t) - P^{(n)}(t) - (n!)M(t-\alpha)^0 = f^{(n)}(t) + 0 - (n!)M \bullet 1 =$ (✳)

So then $P(\alpha) = f(\alpha)$, $P'(\alpha) = f'(\alpha)$, ..., $P^{(n-1)}(\alpha) = f^{(n-1)}(\alpha)$.

And $g(\alpha) = f(\alpha) - P(\alpha) - M(\alpha - \alpha)^n = f(\alpha) - f(\alpha) = 0$,

$g'(\alpha) = f'(\alpha) - P'(\alpha) - (M(\alpha - \alpha)^n)' = f'(\alpha) - f'(\alpha) = 0$, ...,

$g^{(n-1)}(\alpha) = f^{(n-1)}(\alpha) - P^{(n-1)}(\alpha) - (M(\alpha - \alpha)^n)^{(n-1)} = f^{(n-1)}(\alpha) - f^{(n-1)}(\alpha) = 0$.

Note that we also have $g(\beta) = f(\beta) - P(\beta) - \dfrac{f(\beta) - P(\beta)}{(\beta - \alpha)^n}(\beta - \alpha)^n = 0$.

So then $g(\alpha) = g(\beta) = 0 \Rightarrow \exists\, x_1 \ni \alpha < x_1 < \beta$ and $g'(x_1) = 0$.

$g'(\alpha) = g'(x_1) = 0 \Rightarrow \exists\, x_2 \ni \alpha < x_2 < x_1$ and $g''(x_2) = 0$.

⋮

$g^{(n-1)}(\alpha) = g^{(n-1)}(x_{n-1}) = 0 \Rightarrow \exists\, x_n \ni \alpha < x_n < \cdots$ and $g^{(n)}(x_n) = 0$.

From (✳), we get that $0 = g^{(n)}(x_n) = f^{(n)}(x_n) - Mn!$, thus $M = f^{(n)}(x_n)/n!$.

$\therefore\ f(\beta) = P(\beta) + \dfrac{f^{(n)}(x_n)}{n!}(\beta - \alpha)^n$.

Definition For $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$,

$$\mathbf{f}'(x) = \left\langle \lim_{t\to x}\frac{f_1(t)-f_1(x)}{t-x}, \lim_{t\to x}\frac{f_2(t)-f_2(x)}{t-x}, \dots, \lim_{t\to x}\frac{f_n(t)-f_n(x)}{t-x} \right\rangle$$

Theorem **5.19** Let $\mathbf{f}: [a, b] \to \mathbb{R}^k$ be a vector valued continuous function (curve in $\mathbb{R}^k$). If $\mathbf{f}$ is differntiable in (a, b), then $\exists\, x \in (a, b) \ni |\mathbf{f}(b) - \mathbf{f}(a)| \le (b - a)|\mathbf{f}'(x)|$.

Proof:
Define $\varphi:[a, b] \to \mathbb{R}$ by
$\varphi(t) = (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(t) = (f_1(b) - f_1(a))f_1(t) + (f_2(b) - f_2(a))f_2(t) + \cdots$
Then $\exists\, x \in (a, b) \ni \varphi(b) - \varphi(a) = (b - a)\varphi'(x)$.
By definition of φ, $\varphi(b) = (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(b)$ and $\varphi(a) = (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(a)$.
And $\varphi'(x) = (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}'(x)$.
So

$$\begin{aligned}(b-a)(\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}'(x) &= (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(b) - (\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}(a).\\ &= (\mathbf{f}(b) - \mathbf{f}(a)) \bullet (\mathbf{f}(b) - \mathbf{f}(a)).\\ &= |\mathbf{f}(b) - \mathbf{f}(a)|^2.\end{aligned}$$

So then by the Schwartz Inequality,

$$\begin{aligned}|\mathbf{f}(b) - \mathbf{f}(a)|^2 &= (b-a)(\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}'(x)\\ &\le (b-a)|(\mathbf{f}(b) - \mathbf{f}(a)) \bullet \mathbf{f}'(x)|\\ &\le (b-a)|(\mathbf{f}(b) - \mathbf{f}(a))| \bullet |\mathbf{f}'(x)|.\end{aligned}$$

$\therefore\ |\mathbf{f}(b) - \mathbf{f}(a)| \le (b - a)|\mathbf{f}'(x)|$.

Homework Discussion **Week 1, #11:** Let $f: [a, b] \to \mathbb{R}$ be a continuous function.
Define $g: [a, b] \to \mathbb{R}$ by $g(x) = \begin{cases} f(a) : x = a \\ \max\limits_{a\le t\le x} f(t) : a < x \le b \end{cases}$. Prove that g is continuous on $[a, b]$.

Proof:
Fix $x_0 \in [a, b]$.

Case 1: If $f(x_0) < g(x_0)$, then $\exists\, \delta > 0 \ni f(x) < g(x_0)\ \forall\, x \in (x_0 - \delta, x_0 + \delta)$.
Let $t_0 = \sup\{x \in (x_0 - \delta, x_0 + \delta): f(x) = g(x_0)\}$. Then $t_0 < x_0$
$\forall\, x \ni t_0 < x < x_0 + \delta, f(x) < g(x) = g(x_0)$.

Case 2: $f(x_0) = g(x_0)$. Suppose g is not continuous at x_0.
g is increasing, so g has left and right limits at x_0.
Thus $g(x_0-) < g(x_0)$ or $g(x_0) < g(x_0+)$.

Case 2.1: Assume $g(x_0-) < g(x_0)$. Let $\varepsilon = g(x_0) - g(x_0-)$.
For $x < x_0$, $f(x) < g(x) \le g(x_0-) = g(x_0) - \varepsilon = f(x_0) - \varepsilon$.
So $f(x_0-) \le f(x_0) - \varepsilon$, contrary to continuity of f.

For Case 2.2: Assume $g(x_0) < g(x_0+)$. Let $\varepsilon = g(x_0+) - g(x_0)$.
For $x > x_0$, $f(x) \le g(x)$ and $g(x) \ge g(x_0)$.
For $t_n = x_0 + 1/n$, $\exists\, x_0 < t_n \le x_0 + 1/n \ni f(t_n) \ge f(x_0) + \varepsilon/2$, contrary to continuity of f.