Content:

Chapter 6, The Riemann-Stieltjes Integrals

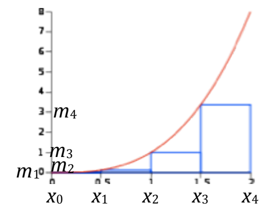
Definitions **6.1** Suppose $f : [a, b] \to \mathbb{R}$ is bounded.

A *partition* of $[a, b]$, $P = \{a = x_0 \le x_1 \le \ldots \le x_n = b\}$.

Define $\Delta x_i = x_i - x_{i-1}$, $M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$, $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$.

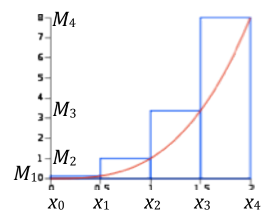
Lower Riemann sum, $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$.

Upper Riemann sum, $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$.



Example $L(P, f) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + m_4\Delta x_4$

$U(P, f) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + M_4\Delta x_4$



Definition **6.3** A partition, P^*, is a refinement of a partition, P, if $P \subset P^*$.

$P^* = \{a = x_0 \le x_1 \le \cdots \le x_{i-1} < x^* < x_i \le \cdots \le x_n = b\}$.

Note f bounded $\Rightarrow \exists\, m, M \in \mathbb{R} \ni m \le f(x) \le M \;\forall\; a \le x \le b$. M_i exists $\Leftrightarrow M_i < +\infty$.

$L(P, f) \le U(P, f)$.

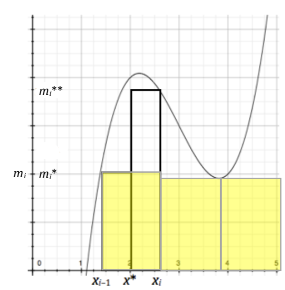
$m(b-a) \le L(P, f) \le L(P^*, f) \le U(P^*, f) \le U(P, f) \le M(b-a)$.

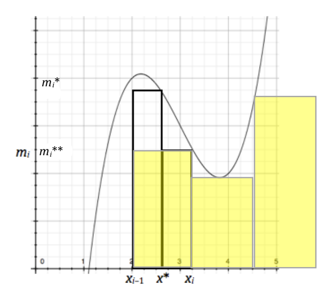
Proof:

Note that

$m_i \le m_i^* = \inf_{x_{i-1} \le x \le x^*} f(x)$ and

$m_i \le m_i^{**} = \inf_{x^* \le x \le x_i} f(x)$.





So then $L(P^*, f)$

$= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_i^*(x^* - x_{i-1}) + m_i^{**}(x_i - x^*) + \cdots + m_n(x_n - x_{n-1})$

$\ge m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_i(x^* - x_{i-1}) + m_i(x_i - x^*) + \cdots + m_n(x_n - x_{n-1})$

$= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_i(x_i - x_{i-1}) + \cdots + m_n(x_n - x_{n-1}) = L(P, f)$.

Similarly $U(P^*, f) \le U(P, f)$.

Thus, $m(b-a) \le L(P, f) \le L(P^*, f) \le U(P^*, f) \le U(P, f) \le M(b-a)$.

For $A = \{L(P, f) : P$ is a partition of $[a, b]\}$, $\sup A = \sup_P L(P,f) = \underline{\int_a^b} f(x)dx$.

For $B = \{U(P, f) : P$ is a partition of $[a, b]\}$, $\inf B = \inf_P U(P,f) = \overline{\int_a^b} f(x)dx$.

We know sup A and inf B exist as A is bounded above by $M(b - a)$ and B is bounded below by $m(b - a)$.
sup $L(P, f)$ and inf $U(P, f)$ are called the *lower and upper Riemann integrals* of f over $[a, b]$.

A bounded function $f: [a, b] \to \mathbb{R}$ is called *Riemann Integrable* $\Leftrightarrow$ $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$.

Note It is evident that $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$ always.

Note If f is continuous on $[a, b]$ then f is Riemann integrable.

Proof:
Let $P = \{a = x_0, x_1, ..., x_n = b\}$.
f is continuous $\Rightarrow \exists\, \delta > 0 \ni |x_i - x_{i-1}| < \delta \Rightarrow |M_i - m_i| < \varepsilon\ \forall\, i \in \{1, 2, .., n\}$.
We can define $P^* \subset P \ni \forall\, i$, $|x_i - x_{i-1}| < \delta$.
This gives us that $\sup_P L(P,f) = \inf_P U(P,f)$.
$\therefore f$ is Riemann integrable.

Note If f is monotone on $[a, b]$ then f is Riemann integrable.

Example For $f: [0, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 2 : x \in Q \cap [0,1] \\ 3 : x \in [0,1] \setminus Q \end{cases}$,
$L(P, f) = 2(x_1 - x_0) + 2(x_2 - x_1) + \cdots + 2(x_n - x_{n-1}) = 2(1 - 0) = 2$.
$U(P, f) = 3(x_1 - x_0) + 3(x_2 - x_1) + \cdots + 3(x_n - x_{n-1}) = 3(1 - 0) = 3$.

We need only that the function is at most measure 0 in order to compute Riemann sums.

Definition A set $S \subset \mathbb{R}$ has measure 0 if $\forall\, \varepsilon > 0\ \exists$ a covering $\{(x_n, y_n) |\, n \in \mathbb{N}\}$ of S such that $\sum_{i=1}^{\infty} (y_n - x_n) < \varepsilon$.

Example Let $S = \{s_1, s_2, ...\}$. We know $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Thus $\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$.
For each i, $s_i \in (s_i - \varepsilon/2^i, s_i + \varepsilon/2^i)$. $\therefore$ S has measure 0.

Riemann-Stieltjes case

Definition **6.2** Let $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing and bounded.
For each partition, P, replace Δx_i by $\Delta\alpha_i$ where $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.
This preserves monotone increasing of x_i.

$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$ and $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ are the upper and lower Riemann Stieltjes sums.

By theorem 6.4, following, we can define

$\sup_P L(P, f, \alpha) = \underline{\int_a^b} f d\alpha = \underline{\int_a^b} f(x) d\alpha(x) < +\infty$, and

$\inf_P U(P, f, \alpha) = \overline{\int_a^b} f d\alpha = \overline{\int_a^b} f(x) d\alpha(x) > -\infty$.

f is Riemann-Stieltjes $\Leftrightarrow \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$.

$\int_a^b f d\alpha$ is called the Riemann-Stieltjes integral.

$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) d(x)$ when α is differential.

Note $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ emphasizes or deemphasizes certain intervals.
It enlarges or shrinks intervals. For example, a function can behave badly on [0, 1/3], but behave nicely on [1/3, 1], yet α can shrink [0, 1/3] and enlarge [1/3, 1] so that we can still integrate over the entire domain.

Theorem **6.4** If P^* is a refinement of P, then
$m(\alpha(b) - \alpha(a)) \le L(P, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P, f, \alpha) \le M(\alpha(b) - \alpha(a))$.

Proof:
Identical to Riemann case above.