Content:

Theorem 4.17 Let X, Y be metric spaces, X compact, $f : X \to Y$ a bijective and continuous mapping. Then $f^{-1}(f(X))$ is a continuous mapping of Y into X.

Definition 4.18 Uniformly Continuous

Theorem 4.19 If X, Y are metric spaces, X compact, and f is continuous, then f is uniformly continuous.

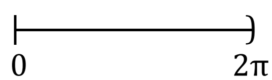
Definition Lipschitz function

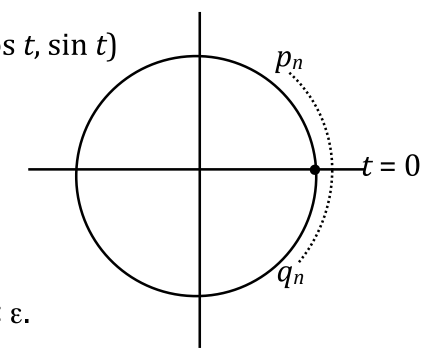
Theorem 4.20 Let E be a noncompact set in $\mathbb{R}^1$. Then (a) $\exists f$, continuous on E, unbounded; (b) $\exists f$, continuous and bounded on E, which has no maximum. If E is bounded, then (c) $\exists f$, continuous on E but not uniformly continuous.

Theorem **4.17** Let X, Y be metric spaces, X compact, $f : X \to Y$ a bijective and continuous mapping. Then $f^{-1}(f(X))$ is a continuous mapping of Y into X.

Proof:
By Theorem 4.8 (*Let X, Y be metric spaces. Then $f : X \to Y$ is a continuous function $\Leftrightarrow f^{-1}(V)$ is open for every open set, $V \subset Y$.*), it suffices to show U open $\Rightarrow f(U)$ is open.
$Y \setminus V = Y \setminus f(U) = f(X \setminus U)$. Since U is open, then $X \setminus U$ is closed.
As X is compact and f is continuous, then $f(X \setminus U)$ is compact (Thm 4.13).
Thus $Y \setminus V$ is closed in Y, hence V is open in Y.

Example **(1)** 4.21 $f : [0, 2\pi) \to S^1$ defined by $f(t) = (\cos t, \sin t)$





f is continuous as
$\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni |x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$.
But f^{-1} is not continuous as,
$p_n \to (1, 0)$ and $q_n \to (1, 0)$, but $f^{-1}(p_n) \to 0$ and $f^{-1}(q_n) \to 2\pi$.

(2) Define $f : [0, 1] \to [0, 1]$ by $f(x) = x$.
Define the metric on the domain as the discrete metric and the metric on the codomain as the standard metric $d(x, y) = |x - y|$.
Note that for $x_n \subset \text{dom}(f)$, $x_n \to x_0 \Leftrightarrow x_n = x_0$ for each $n \in \mathbb{N}$.
f is continuous as $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0) = x_0$.
But f^{-1} is not continuous as,
$\forall\, y_n \in f[0, 1] \ni y_n \neq y\ \forall\, n \in \mathbb{N}, y_n \to y \not\Rightarrow f^{-1}(y_n) \to f^{-1}(y)$.

Note The examples above demonstrate the need for compactness of the domain. [0, 1] is not compact in (2) as $B(x, 1/2)_{x \in [0, 1]}$ is an open cover of [0, 1], but there is no finite subcover.

Definition **4.18** For X, Y metric spaces, $f : X \to Y$ is *uniformly continuous* if $\forall\, \varepsilon > 0, \exists\, \delta = \delta(\varepsilon) \ni d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.
For this δ, $\delta(\varepsilon) = \inf \delta(\varepsilon, x_0)_{x_0 \in X}$.

Example **(1)** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. To show f is continuous, let $\varepsilon < 0$ and let $x_0 \in \mathbb{R}$. Note that

$$|x - x_0| < \frac{x_0}{2} \Rightarrow -\frac{x_0}{2} < x - x_0 < \frac{x_0}{2} \Rightarrow \frac{3x_0}{2} = -\frac{x_0}{2} + 2x_0 < x + x_0$$
$$< \frac{x_0}{2} + 2x_0 = \frac{5x_0}{2}.$$ Choose $\delta = \frac{2\varepsilon}{5x_0}$. Then
$$|x^2 - x_0^2| = |x - x_0|\,|x + x_0| < \frac{2\varepsilon}{5x_0}|x + x_0| < \frac{2\varepsilon}{5x_0} \bullet \frac{5x_0}{2} = \varepsilon.$$
However, f is not uniformly continuous.

(2) Define $f : \mathbb{R} \to (-\pi/2, \pi/2)$ by $f(x) = \tan^{-1} x$.
f is uniformly continuous.

Definition $f : X \to Y$ is called Lipschitz if $\exists\, L \geq 0 \ni d(f(x), f(y)) \leq L d(x, y)$
So then if we want $d(f(x), f(y)) < \varepsilon$, we need $d(x, y) < \delta = \varepsilon/L$.
Note that if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, then $|f'(x)| < L$ as
$d(f(x), f(y)) \leq L d(x, y) \Rightarrow f'(x) = d(f(x), f(y))/d(x, y) \leq \text{L}$.

Example $f : [0, 1] \to [0, 1]$ defined by $f(x) = \sqrt{x}$ is not Lipschitz as its derivative is ∞ at $x = 0$.

Note If f is Lipschitz, then it is uniformly continuous.
If f is uniformly convergent, we can see by the above example that f may not be Lipschitz.

Theorem **4.19** If X is a compact metric space, Y is a metric space, and f is continuous, then f is uniformly continuous.

Proof:
Let $x_0 \in X$ be arbitrary.
Fix $\varepsilon > 0$. Then $\exists\, \delta = \delta(\varepsilon) \ni d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.
Consider the open cover of X by the balls $B(x_\alpha, \frac{1}{2}\delta(\varepsilon, x_\alpha))$.
By compactness of X we can select a finite subcover
$B(x_1, \frac{1}{2}\delta(\varepsilon, x_1)), \ldots, B(x_n, \frac{1}{2}\delta(\varepsilon, x_n))$. Choose $\delta = \frac{1}{2}\min\{\delta(\varepsilon, x_1), \ldots, \delta(\varepsilon, x_n)\}$.
Let $x, y \in X \ni d(x, y) < \delta$. Then $\exists\, x_{n_0} \in X \ni d(x, x_{n_0}) < \frac{1}{2}\delta_{n_0}$.
$d(x_{n_0}, y) \leq d(x_{n_0}, x) + d(x, y) < \frac{1}{2}\delta_{n_0} + \frac{1}{2}\delta_{n_0} = \delta_{n_0}$.
$d(f(x), f(y)) \leq d(f(x), f(x_{n_0})) + d(f(x_{n_0}), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
$\therefore$ f is uniformly continuous.

Exercise Consider $f: [0, 1] \to [0, 1]$ defined by $f(x) = \sqrt{x}$. Let $\varepsilon > 0$.
Construct $\delta(\varepsilon)$ to show f is uniformly continuous.
Hint: Use $\left|\sqrt{x} - \sqrt{x_0}\right| = \left|\dfrac{x - x_0}{\sqrt{x} - \sqrt{x_0}}\right|$.

Homework #7, Change to [0, 1] as compactness gives uniform continuity.

Theorem **4.20** Let E be a noncompact set in $\mathbb{R}^1$. Then
(a) $\exists f$, continuous on E, not bounded.
(b) $\exists f$, continuous and bounded on E, which has no maximum.
If, in addition, E is bounded, then
(c) $\exists f$, continuous on E which is not uniformly continuous.

Proof:
For E unbounded,
(a) $f(x) = x$ is continuous and not bounded.
For E bounded,
(a) $f(x) = \dfrac{1}{x - x_0}$ where $x_0 \in E' \backslash E$ is continuous and unbounded.
For E unbounded,
(b) $f(x) = \dfrac{x^2}{1 + x^2}$ is continuous, bounded, and has no maximum.
For E bounded,
(b) $f(x) = \dfrac{1}{1 + (x - x_0)^2}$ is continuous, bounded, and has no maximum.
For E bounded,
(c) $f(x) = \dfrac{1}{x - x_0}$ is not uniformly continuous.