Content:

Theorem 6.5 $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$.

Theorem 6.6 If $f \in \mathscr{R}(\alpha)$ on $[a, b] \Leftrightarrow \forall\, \varepsilon > 0, \exists\, P_\varepsilon \ni U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$.

Theorem 6.7 (a) If $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$ (✳) for some P_ε and some ε, then (✳) for $P^* \supset P_\varepsilon$; (b) If (✳) for $P_\varepsilon = \{a = x_0 < x_1 < ..., x_n = b\}$ and $s_i, t_i \in [x_{i-1}, x_i]$, (✳✳) then $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon$; (c) If $f \in \mathscr{R}(\alpha)$ and (✳✳) then $\left|\sum_{i=1}^n f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \varepsilon$.

Theorem 6.8 If $f \in C[a, b]$, then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

Theorem 6.9 If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

Summary Let $\alpha : [a, b] \to \mathbb{R}$ be monotone increasing and bounded.
Choose a partition, $P = \{a = x_0 \le x_1 \le ... \le x_n = b\}$.
For $M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$, $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$, and $\Delta\alpha_i$ where $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$,
Lower Riemann Stieltjes sum = $L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$ and
Upper Riemann Stieltjes sum = $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$.
f is Riemann integrable with respect to α if
$\sup_P L(P, f, \alpha) = \inf_P U(P, f, \alpha)$, that is, $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$.
$\int_a^b f d\alpha$ is called the Riemann-Stieltjes integral.
We say $f \in \mathscr{R}(\alpha)$.

Theorem **6.4** If P^* is a refinement of P, then
$m(\alpha(b) - \alpha(a)) \le L(P, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P, f, \alpha) \le M(\alpha(b) - \alpha(a))$.

Theorem **6.5** $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$.

Proof:
Let P be a fixed partition. Let P^* be a refinement of P.
From Theorem 6.4, we know $L(P, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P, f, \alpha)$.
Since $L(P, f, \alpha) \le U(P, f, \alpha) \le \inf_{P^*} U(P^*, f, \alpha) = \overline{\int_a^b} f d\alpha$ and
$\underline{\int_a^b} f d\alpha = \sup_P L(P, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha)$, then $\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$.

Theorem **6.6** If $f \in \mathscr{R}(\alpha)$ on $[a, b] \Leftrightarrow \forall\, \varepsilon > 0, \exists\, P_\varepsilon \ni U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$.

Proof:

$\Rightarrow$: $f \in \mathscr{R}(\alpha)$ on $[a, b] \Rightarrow \sup\limits_P L(P, f, \alpha) = \int_a^b f d\alpha = \inf\limits_P U(P, f, \alpha)$.

Let $\varepsilon > 0$. Then $\exists\, Q_\varepsilon \ni \int_a^b f d\alpha - L(Q_\varepsilon, f, \alpha) < \varepsilon/2$.

And $\exists\, R_\varepsilon \ni U(R_\varepsilon, f, \alpha) - \int_a^b f d\alpha < \varepsilon/2$. Thus $U(R_\varepsilon, f, \alpha) - L(Q_\varepsilon, f, \alpha) < \varepsilon$.

Choose $P_\varepsilon = Q_\varepsilon \cup R_\varepsilon$. Then, by Theorem 6.4, $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$.

$\Leftarrow$: Assume $\forall\, \varepsilon > 0, \exists\, P_\varepsilon \ni U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$.

Then we have $L(P_\varepsilon, f, \alpha) \le U(P_\varepsilon, f, \alpha) < L(P_\varepsilon, f, \alpha) + \varepsilon$

$\Rightarrow L(P_\varepsilon, f, \alpha) \le \sup\limits_{P_\varepsilon \subset P^*} L(P^*, f, \alpha) \le U(P_\varepsilon, f, \alpha) < L(P_\varepsilon, f, \alpha) + \varepsilon \le \sup\limits_{P_\varepsilon \subset P^*} L(P^*, f, \alpha) + \varepsilon$

$\Rightarrow \underline{\int_a^b} f d\alpha \le U(P_\varepsilon, f, \alpha) < \underline{\int_a^b} f d\alpha + \varepsilon$

$\Rightarrow \underline{\int_a^b} f d\alpha \le \inf\limits_{P_\varepsilon \subset P^*} U(P^*, f, \alpha) = \overline{\int_a^b} f d\alpha < \underline{\int_a^b} f d\alpha + \varepsilon$

$\Rightarrow \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$. $\therefore f \in \mathscr{R}(\alpha)$.

Theorem **6.7** **(a)** If $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$ for some P_ε and some ε, then $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$ for any refinement of P_ε.

(b) If $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$ for $P_\varepsilon = \{a = x_0 < x_1 < \ldots, x_n = b\}$ and $\forall\, i = 1, 2, \ldots, n$, $s_i, t_i \in [x_{i-1}, x_i]$, then $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon$.

(c) If $f \in \mathscr{R}(\alpha)$, $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$ for $P_\varepsilon = \{a = x_0 < x_1 < \ldots, x_n = b\}$ and $t_i \in [x_{i-1}, x_i]$, then $\left| \sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon$.

Proof:

(a) By Theorem 6.4,

$L(P_\varepsilon, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P_\varepsilon, f, \alpha)$ for any refinement of P^*.

Thus $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon \Rightarrow L(P^*, f, \alpha) - U(P^*, f, \alpha) < \varepsilon$.

(b) Case 1: $f(s_i) \le f(t_i) \Rightarrow m_i \le f(s_i) \le f(t_i) \le M_i$.

So then $f(t_i) - f(s_i) \le M_i - f(s_i) \le M_i - m_i$.

Case 2: $f(t_i) \le f(s_i) \Rightarrow f(s_i) - f(t_i) \le M_i - m_i$, or equivalently

$-(M_i - m_i) \le f(t_i) - f(s_i)$.

Thus, $|f(t_i) - f(s_i)| \le M_i - m_i$. So then $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i \le$

$\sum_{i=1}^n (M_i - m_i) \Delta\alpha_i = \sum_{i=1}^n M_i \Delta\alpha_i - \sum_{i=1}^n m_i \Delta\alpha_i = U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$.

Theorem **6.7 (c)** If $f \in \mathscr{R}(\alpha)$, $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$ for $P_\varepsilon = \{a = x_0 < x_1 < ..., x_n = b\}$ and $t_i \in [x_i - x_{i-1}]$, then $\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \varepsilon$.

(c) $m_i \le f(t_i) \le M_i \Rightarrow$
$L(P_\varepsilon, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i = U(P_\varepsilon, f, \alpha)$.
Since $f \in \mathscr{R}(\alpha)$, then $L(P_\varepsilon, f, \alpha) \le \int_a^b f d\alpha \le U(P_\varepsilon, f, \alpha)$.
So then $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon \Rightarrow$
$0 < U(P_\varepsilon, f, \alpha) - \sum_{i=1}^{n} f(t_i)\Delta\alpha_i < U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$ and
$0 < U(P_\varepsilon, f, \alpha) - \int_a^b f d\alpha < U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$. Thus
$-\varepsilon < \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - U(P_\varepsilon, f, \alpha) < \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha < U(P_\varepsilon, f, \alpha) - \int_a^b f d\alpha < \varepsilon$.
$\therefore$ $\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \varepsilon$.

Theorem **6.8** If $f \in C[a, b]$, then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

Proof:
Let $\varepsilon > 0$ be fixed, but arbitrary. By the uniform continuity of f on $[a, b]$,
$\exists\ \delta > 0 \ni |s - t| < \delta \Rightarrow |f(s) - f(t)| < \dfrac{\varepsilon}{2(\alpha(b) - \alpha(a))}$.
Choose $P_\varepsilon \ni \Delta x_i < \delta\ \forall\ i \in \{1, 2, ..., n\}$.
$\therefore$ $M_i - m_i \le \dfrac{\varepsilon}{2(\alpha(b) - \alpha(a))}$, $i \in \{1, 2, ..., n\}$. Thus $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) =$
$\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i < \dfrac{\varepsilon}{2(\alpha(b) - \alpha(a))} \sum_{i=1}^{n} \Delta\alpha_i = \dfrac{\varepsilon}{2(\alpha(b) - \alpha(a))} \bullet (\alpha(b) - \alpha(a)) = \varepsilon$.
$\therefore$ By Theorem 6.6 above, $f \in \mathscr{R}(\alpha)$.

Theorem **6.9** If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

Proof:
Let $\varepsilon > 0$.
α is continuous on $[a, b] \Rightarrow \alpha$ is uniformly continuous.
$\therefore \exists\, \delta > 0 \ni |x - y| < \delta \Rightarrow |\alpha(x) - \alpha(y)| < \dfrac{\varepsilon}{f(b) - f(a)}$.
Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\} \ni \Delta x_i < \delta$.
Suppose f is monotonically increasing. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$.
Thus $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \sum_{i=1}^{n}\big(f(x_i) - f(x_{i-1})\big)\Delta\alpha_i$
$= [f(x_1) - f(x_0)][\alpha(x_1) - \alpha(x_0)] + \cdots + [f(x_n) - f(x_{n-1})][\alpha(x_n) - \alpha(x_{n-1})]$
$\leq [f(x_1) - f(x_0)]\dfrac{\varepsilon}{f(b) - f(a)} + \cdots + [f(x_n) - f(x_{n-1})]\dfrac{\varepsilon}{f(b) - f(a)}$
$= \dfrac{\varepsilon}{f(b) - f(a)}[f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})]$
$= \dfrac{\varepsilon}{f(b) - f(a)}[f(b) - f(a)] = \varepsilon$.