Content:

Theorem 6.10 Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Theorem 6.11 Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.

Theorem 6.12 (a) If $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ on $[a, b]$, then $f_1 + f_2 \in \mathscr{R}(\alpha)$, and $\int_a^b (f_1 + f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$; if $c \in \mathbb{R}$, then $cf \in \mathscr{R}(\alpha)$ and $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$.

(e) If $f \in \mathscr{R}(\alpha_1)$, $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$, $\int_a^b f\,d(\alpha_1 + \alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$;

if $f \in \mathscr{R}(\alpha)$ and $c \in \mathbb{R}^+$, then $f \in \mathscr{R}(c\alpha)$ and $\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$.

Theorem **6.10** Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Proof:

Let $\varepsilon > 0$. Since f is bounded, $\exists\, M = \sup_{[a,b]}|f(x)|$.

Let $\{t_1, \ldots, t_k\}$ be the set of points of discontinuity of f.

Since α is continuous at every point of discontinuity,

$\exists\, \delta > 0 \ni |x - t_i| < \delta \Rightarrow \alpha(x) - \alpha(t_i) < \varepsilon/(4kM)$.

Thus, $\exists\, [u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]$, disjoint intervals such that for each i, $t_i \in (u_i, v_i)$. Let $E = [a, b] \setminus \bigcup_{j=1}^{k}(u_j, v_j)$.

E is compact as it is the complement of an open set.

Since E does not contain any of the points of discontinuity of f, then $f(E)$ is continuous. Hence $f(E)$ is uniformly continuous.

So then $\exists\, \delta_1 > 0 \ni \delta_1 < \delta$ and $\forall\, x, y \in E$ where $|x - y| < \delta_1$, we have

$|f(x) - f(y)| < \dfrac{\varepsilon}{2[\alpha(b) - \alpha(a)]}$.

Let $P_\varepsilon = \{a = x_0, x_1, \ldots, x_n = b\} \cup \{u_1, v_1, \ldots, u_k, v_k\}$ where $x_i \notin (u_j, v_j)$ $\forall\, i \in \{1, 2, \ldots, n\}$ and $\forall\, j \in \{1, 2, \ldots, k\}$.

Relabel this partition as $\{a = y_0, y_1, \ldots, y_r = b\}$.

Let $A = \{i: y_i = v_j \text{ for some } i, j\}$. Let $B = \{P_\varepsilon \setminus A\}$

Note that $M_i - m_i \le 2M$, $\forall\, i \in A$, and $M_i - m_i \le \varepsilon$, $\forall\, i \in B$.

So then we have $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) =$

$$\sum_{i \in B}(M_i - m_i)\Delta\alpha_i + \sum_{i \in A}(M_i - m_i)\Delta\alpha_i <$$

$$k \bullet 2M \bullet \frac{\varepsilon}{4kM} + \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]} \bullet [\alpha(b) - \alpha(a)] = \varepsilon. \quad \therefore\ f \in \mathscr{R}(\alpha).$$

Note If $\alpha(x) = x$, then $f \in \mathscr{R}(\alpha)$ if the discontinuity set of f has measure 0.

Theorem **6.11** Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.

Proof:
Let $\varepsilon > 0$.
By continuity of ϕ on $[m, M]$, hence uniform continuity,
$\exists\ \delta > 0 \ni \delta < \varepsilon$ and $\forall\ t, s \in [m, M]$ where $|t - s| < \delta$, $|\phi(t) - \phi(s)| < \varepsilon$.
Note that $f([a, b]) \subset [m, M]$.
$f \in \mathscr{R}(\alpha) \Rightarrow \exists\ Q_\varepsilon \ni U(Q_\varepsilon, f, \alpha) - L(Q_\varepsilon, f, \alpha) < \delta^2$.
Let $M_i^* = \sup\limits_{x_{i-1} \le x \le x_i} h(x)$, let $m_i^* = \inf\limits_{x_{i-1} \le x \le x_i} h(x)$, and let $M^* = \sup\limits_{m \le t \le M} |\phi(t)|$.
Let $A = \{i: M_i - m_i \le \delta\}$, and let $B = \{i: \delta < M_i - m_i\}$.
So if $i \in A$, $|\phi(M_i) - \phi(m_i)| < \varepsilon$, hence $M_i^* - m_i^* < \varepsilon$,
and if $i \in B$, then $\delta < M_i - m_i$ which implies
$\sum\limits_{i \in B} \delta \Delta\alpha_i < \sum\limits_{i \in B} (M_i - m_i)\Delta\alpha_i = U(Q_\varepsilon, f, \alpha) - L(Q_\varepsilon, f, \alpha) < \delta^2$. Thus $\sum\limits_{i \in B} \Delta\alpha_i < \delta$.
So then $U(Q_\varepsilon, h, \alpha) - L(Q_\varepsilon, h, \alpha) = \sum\limits_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum\limits_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i$
$\le \varepsilon(\alpha(b) - \alpha(a)) + 2M^*\delta < \varepsilon(\alpha(b) - \alpha(a) + 2M^*)$.
As ε is arbitrary, then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.

Theorem **6.12** **(a)** If $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ on $[a, b]$, then $f_1 + f_2 \in \mathscr{R}(\alpha)$,
and $\int_a^b (f_1 + f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$;
if $c \in \mathbb{R}$, then $cf \in \mathscr{R}(\alpha)$ and $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$.

(e) If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f\,d(\alpha_1 + \alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2;$$

if $f \in \mathscr{R}(\alpha)$ and c is a positive constant, then
$f \in \mathscr{R}(c\alpha)$ and $\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$.

Note The above property is known as the Riemann-Stieltjes Integral linearity.