Content:

Theorem 6.17 Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$.
Let f be a bounded real function on $[a, b]$. Then $f \in \mathscr{R}(\alpha) \Leftrightarrow f' \in \mathscr{R}$.
In that case $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

Homework Discussion

Week 7, #2: Suppose that:
α is differentiable on $[a, b]\backslash\{c\}$
$\alpha' \in \mathscr{R}$ on $[a, c - \varepsilon]$ and $[c + \varepsilon, b]$ for any small $\varepsilon > 0$
$A = \int_a^c f(x)\alpha'(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\alpha'(x)dx$ exists
$B = \int_c^b f(x)\alpha'(x)dx = \lim_{\varepsilon \to 0} \int_{c+\varepsilon}^b f(x)\alpha'(x)dx$ exists
f is continuous at c.
Then we have $\int_a^c f d\alpha = A + f(c)(\alpha(c) - \alpha(c-))$
and $\int_c^b f d\alpha = B + f(c)(\alpha(c) - \alpha(c-))$

#3 Change $x - 3$ to $x + 3$ to make it an increasing function
Then $\int_0^2 e^x dx = \int_0^{1-\varepsilon} e^x dx + \int_{1+\varepsilon}^2 2e^x x dx + e = 2e^2 + 2e - 1$

#5 Use $\sum_{n=1}^{\infty} \varepsilon \bullet \frac{1}{2^n}$. Similar to Theorem 6.10.

#6 $f(x) = \begin{cases} \frac{1}{n} : x = r_n \\ 0 : x \notin Q \cap [0, 1] \end{cases}$. Fix $x = r_{n_0}$. Then $f(r_{n_0}) = 1/n_0$, so
the function is discontinuous at all rationals. If $x_k \in \mathbb{Q}$, then $x_k \to r_{n_0}$.
This is a sequence that converges to r_{n_0}. That is, $x_k = r_{n_k}$.
If $\frac{m_k}{n_k} \to \frac{1}{3}$, then n_k cannot be bounded.

7 Similar

9 $f'(x) = -f'(x) \Rightarrow f'(x) \equiv 0 \Rightarrow f$ is constant.
$\int_a^a f(t)dt = \int_a^b f(t)dt = 0$. And $c(b - a) = 0 \Rightarrow c = 0$.

10 Use contradiction. Suppose $g(x_0) > 0$. By continuity of g,
$g(x) \geq g(x_0)/2$. This cannot be 0, so we have a contradiction.

Theorem **6.17** Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$.
Let f be a bounded real function on $[a, b]$. Then $f \in \mathscr{R}(\alpha) \Leftrightarrow f\alpha' \in \mathscr{R}$.
In that case $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

Proof:
Let $\varepsilon > 0$. Let $M = \sup_{a \le x \le b} |f(x)|$.
Since $\alpha' \in \mathscr{R}$, then by Thm 6.6 $\exists\, P = \{x_0, ..., x_n\}$ of $[a, b] \ni U(P, \alpha') - L(P, \alpha') < \varepsilon/M$.
By the mean value theorem, $\exists\, t_i \in [x_{i-1}, x_i] \ni \dfrac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(t_i)$, that is
$\Delta\alpha_i = \alpha'(t_i)\Delta x_i$ for $i \in \{1, 2, ..., n\}$.
And by Theorem 6.7 (b), $\forall\, s_i \in [x_{i-1}, x_i]$, $\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \varepsilon/M$.
Since $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$ for $i \in \{1, 2, ..., n\}$, then
$\sum_{i=1}^n f(s_i)\Delta\alpha_i = \sum_{i=1}^n f(s_i)\alpha'(t_i)\Delta x_i$. Since $\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \varepsilon/M$, then
$$\left|\sum_{i=1}^n f(s_i)\Delta\alpha_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i\right| = \left|\sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i - \sum_{i=1}^n f(s_i)\alpha'(t_i)\Delta x_i\right|$$
$$= \sum_{i=1}^n |f(s_i)||\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \le \sum_{i=1}^n M|\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < M \bullet \varepsilon/M = \varepsilon.$$
$\therefore\ -\varepsilon + \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i < \sum_{i=1}^n f(s_i)\Delta\alpha_i < \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i + \varepsilon$. Thus,
$\sum_{i=1}^n f(s_i)\Delta\alpha_i \le \sum_{i=1}^n M^*_i\, \Delta x_i + \varepsilon = U(P, f\alpha') + \varepsilon$.
Since this is true $\forall\, s_i \in [x_{i-1}, x_i]$, $\forall\, i \in \{1, 2, ..., n\}$,
then $U(P, f, \alpha) = \sum_{i=1}^n M_i\Delta\alpha_i \le U(P, f\alpha') + \varepsilon$. Since
$$\left|\sum_{i=1}^n f(s_i)\Delta\alpha_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i\right| = \left|\sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i - \sum_{i=1}^n f(s_i)\Delta\alpha_i\right|$$
then we also have $U(P, f\alpha') \le U(P, f, \alpha) + \varepsilon$. $\therefore$ $|U(P, f, \alpha) - U(P, f\alpha')| \le \varepsilon$.
Since $U(P^*, \alpha') - L(P^*, \alpha') < \varepsilon\ \forall\, P^* \supseteq P$, then $|U(P^*, f, \alpha) - U(P^*, f\alpha')| \le \varepsilon$.
$\therefore\ -\varepsilon - U(P^*, f\alpha') \le U(P^*, f, \alpha) \le U(P^*, f\alpha') + \varepsilon$.
Thus, $-\varepsilon - \overline{\int_a^b} f(x)\alpha'(x)dx \le -\varepsilon - U(P^*, f\alpha') \le U(P^*, f, \alpha)$.
Since this is true for all refinements of P, then $-\varepsilon - \overline{\int_a^b} f(x)\alpha'(x)dx \le \overline{\int_a^b} f d\alpha$.
Since $|U(P^*, f, \alpha) - U(P^*, f\alpha')| = |U(P^*, f\alpha') - U(P^*, f, \alpha)|$, then we also have
$-\varepsilon - \overline{\int_a^b} f d\alpha \le \overline{\int_a^b} f(x)\alpha'(x)dx$. $\therefore$ $\left|\overline{\int_a^b} f d\alpha - \overline{\int_a^b} f(x)\alpha'(x)dx\right| \le \varepsilon$.
Since ε is arbitrary, then $\overline{\int_a^b} f d\alpha = \overline{\int_a^b} f(x)\alpha'(x)dx$.
By similar argument, $\underline{\int_a^b} f d\alpha = \underline{\int_a^b} f(x)\alpha'(x)dx$. $\therefore$ $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.
So then $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha \Leftrightarrow \overline{\int_a^b} f(x)\alpha'(x)dx = \underline{\int_a^b} f(x)\alpha'(x)dx$.
$\therefore$ $f \in \mathscr{R}(\alpha) \Leftrightarrow f\alpha' \in \mathscr{R}$.