Content:

Theorem 6.19 Suppose φ is a strictly increasing continous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then $g \in \mathscr{R}(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Theorem 6.20 Let $f \in \mathscr{R}(\alpha)$ on $[a, b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Theorem 6.21 If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Theorem 6.22 Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then $\int_a^b F(x)g(x)\,dx = F(b)G(b) - \int_a^b f(x)G(x)\,dx$.

Theorem **6.19** Suppose φ is a strictly increasing continous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then $g \in \mathscr{R}(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Proof:

Let $\varepsilon > 0$. Since $f \in \mathscr{R}(\alpha)$, $\exists\, P = \{a = x_0, \ldots, x_n = b\} \ni U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

As φ is strictly increasing and onto, $Q = \{A = \varphi^{-1}(x_0), \ldots, \varphi^{-1}(x_n) = B\}$,

is a partition of $[A, B]$. All partitions of $[A, B]$ can be obtained in this way.

Let $t_i \in [x_{i-1}, x_i]$. Then $f(t_i) = g(\varphi^{-1}(t_i))$ where $\varphi^{-1}(t_i) \in [\varphi^{-1}(x_{i-1}), \varphi^{-1}(x_i)]$.

Thus $U(Q, g, \beta) = U(P, f, \alpha)$ and $L(Q, g, \beta) = L(P, f, \alpha)$.

This gives us that $g \in \mathscr{R}(\beta)$.

By Thm 6.6, $\exists\, P^* \supseteq P \ni U(P^*, f, \alpha) - \int_a^b f\,d\alpha < \varepsilon/2$

and $U(\varphi^{-1}(P^*), g, \beta) - \int_A^B g\,d\beta < \varepsilon/2$, hence $\left|\int_A^B g\,d\beta - \int_a^b f\,d\alpha\right| \le$

$$\left|\int_A^B g\,d\beta - U(\varphi^{-1}(P^*), g, \beta)\right| + \left|U(\varphi^{-1}(P^*), g, \beta) - U(P^*, f, \alpha)\right| + \left|U(P^*, f, \alpha) - \int_a^b f\,d\alpha\right|$$

$< \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon$. $\therefore$ $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Theorem **6.20** Let $f \in \mathscr{R}$ on $[a, b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)dt$. Then

(a) F is continuous on $[a, b]$; and
(b) if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Proof:
(a) Let $\varepsilon > 0$. Let $M = \sup_{a \le x \le b} f(x)$, and let $x, y \in [a, b]$.

By Thm 6.12 (d), $f \in \mathscr{R} \Rightarrow \int_x^y |f(t)|dt \le M|y - x|$. Let $\delta = \dfrac{\varepsilon}{M}$.

Then $|y - x| < \delta \Rightarrow |F(y) - F(x)| = \left|\int_x^y f(t)dt\right| \le \int_x^y |f(t)|dt \le M|y - x| < \varepsilon$.

(b) Assume f is continuous at $x_0 \in [a, b]$. Let $\varepsilon > 0$.
Then $\exists\, \delta > 0 \ni |t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon$.

$$\left|\frac{F(x_0+h) - F(x_0)}{h} - f(x_0)\right| = \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt - f(x_0)\right| =$$

$$\left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)dt\right| \le \left|\frac{1}{h}\right|\left|\int_{x_0}^{x_0+h} |f(t) - f(x_0)|dt\right|.$$

And so $\left|\dfrac{F(x_0+h) - F(x_0)}{h} - f(x_o)\right| < \varepsilon$.

Hence $f(x_0) - \varepsilon < \dfrac{F(x_0+h) - F(x_0)}{h} < f(x_0) + \varepsilon$.

$\Rightarrow \lim_{h \to 0} (f(x_0) - \varepsilon) < \lim_{h \to 0} \dfrac{F(x_0+h) - F(x_0)}{h} < \lim_{h \to 0} (f(x_0) + \varepsilon)$

$\Rightarrow f(x_0) - \varepsilon < F'(x_0) < f(x_0) + \varepsilon \Rightarrow F'(x_0) = f(x_0)$.

Theorem **6.21** (Fundamental Theorem of Calculus)
If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof:
Let $\varepsilon > 0$. Then $f \in \mathscr{R} \Rightarrow \exists\, P = \{x_0, x_1, ..., x_n\} \ni U(P, f) - L(P, f) < \varepsilon$.
Let $i \in \{1, 2, ..., n\}$. Then by continuity of F and the mean value theorem, $\exists\, t_i \in [x_{i-1}, x_i] \ni F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$.

So then $|F(b) - F(a) - \int_a^b f(x)dx|$

$= |F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_1) - F(x_0) - \int_a^b f(x)dx|$

$= |f(t_n)\Delta x_n + f(t_{n-1})\Delta x_{n-1} + \cdots + f(t_1)\Delta x_1 - \int_a^b f(x)dx|$

$= |\sum_{i=1}^n f(t_i)\Delta x_i - \int_a^b f(x)dx| < \varepsilon$ by Theorem 6.7(c).

$\therefore \int_a^b f(x)dx = F(b) - F(a)$.

Theorem **6.22** (Integration by Parts)

Suppose F and G are differentiable functions of $[a, b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then $\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$.

Proof:

By the fundamental theorem of calculus and product rule of derivatives, we have

$$F(b)G(b) - F(a)G(a) = \int_a^b (F(x)G(x))'dx = \int_a^b \left[F(x)G'(x) + F'(x)G(x)\right]dx$$

$$= \int_a^b F(x)G'(x)dx + \int_a^b F'(x)G(x)dx = \int_a^b F(x)g(x)dx + \int_a^b f(x)G(x)dx$$

Thus, $\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$.