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Homework Discussion Week 8, #3

Let $f \in C[0, 1] \ni f$ is differentiable on $(0, 1)$ and $|f'(x)| \le M$, $\forall\, x \in (0, 1)$.

Show that $\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^n f(\frac{k}{n}) \right| \le \frac{M}{n}$, $\forall\, n \in \mathbb{N}$.

Use these facts:

$\exists\, t_0 \in [a, b] \ni \int_a^b f(x)dx = f(t_0)(b - a)$.

$F(x) = \int_a^x f(t)dt \Rightarrow F'(x) = f(x)$.

$F(x_1) - F(x_2) = F'(t_0)(x_1 - x_2)$.

Divide [0, 1] into n intervals.

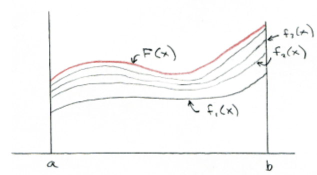
Use the above facts for each interval.

Then $\left| \frac{1}{n}\sum_{k=1}^n \left(f(t_k) - f(\frac{k}{n}) \right) \right| = \left| \frac{1}{n}\sum_{k=1}^n f'(s_k)\left(t_k - \frac{k}{n} \right) \right| \le n \bullet \frac{1}{n} \bullet M \bullet \frac{1}{n}$.

Chapter 7 Many problems from comprehensive exams are from chapter 7.

Definition **7.1** Given $\{f_n(x) |$ for each $n \in \mathbb{N}, f_n(x): E \to \mathbb{R}\}$ we want to calculate

$\lim_{n\to\infty} f_n(x) = f(x)$, the *limit function.*

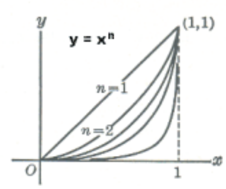


Example Define $f_n(x) = x^n$, $x \in [0, 1]$, $n \in \mathbb{N}$.

$\lim_{n\to\infty} x^n = 0$ for $0 \le x < 1$

$\lim_{n\to\infty} x^n = 1$ for $x = 1$

The limit function is $f(x) = \begin{cases} 0 : 0 \le x < 1 \\ 1 : x = 1 \end{cases}$



Question Is it true in general that $\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x)$?

Example Consider $f_n(x) = x^n$, as defined above.

For $x_0 = 1$, $\lim_{x\to 1}\lim_{n\to\infty} x^n = \lim_{x\to 1-} 0 = 0$.

And $\lim_{n\to\infty}\lim_{x\to 1} x^n = \lim_{n\to\infty} 1^n = 1$.

Question Does $\int_0^1 \lim_{n\to\infty} x^n dx = \lim_{n\to\infty}\int_0^1 x^n dx$?

$\int_0^1 f(x)dx = \int_0^1 0dx = 0$ and $\lim_{n\to\infty}\int_0^1 x^n dx = \lim_{n\to\infty} \left. \frac{x^{n+1}}{n+1} \right|_0^1 = \lim_{n\to\infty} \frac{1}{n+1} = 0$.

Example **7.6** Define $f_n(x) = n^2x(1-x^2)^n$, $x \in [0, 1]$, $n \in \mathbb{N}$.

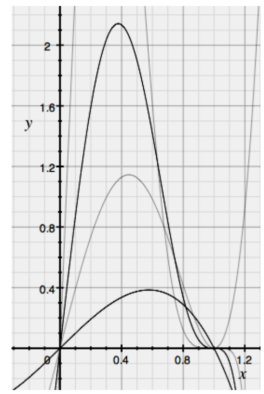
We will show $\int_0^1 f(x) \neq \lim_{n\to\infty} \int_0^1 f_n(x)dx$.

$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} f_n(1) = 0$. And $\forall\, x \in (0, 1)$,

$f(x) = \lim_{n\to\infty} n^2 x(1-x)^n$

$$= \lim_{n\to\infty} \frac{n^2 x}{(1-x^2)^{-n}} = \lim_{n\to\infty} \frac{2nx}{-(1-x^2)^{-n}\ln(1-x^2)} = \lim_{n\to\infty} \frac{2x}{(1-x^2)^{-n}\left[\ln(1-x^2)\right]^2} = 0.$$

And so $\int_0^1 \lim_{n\to\infty} n^2 x(1-x^2)^n\, dx = 0$.



But $\lim_{n\to\infty} \int_0^1 n^2 x(1-x^2)^n\, dx = \lim_{n\to\infty}\left(-\frac{1}{2}\right)\int_0^1 n^2(-2x)(1-x^2)^n\, dx$

$$= \lim_{n\to\infty}\left(\left.\frac{-n^2\left(1-x^2\right)^{n+1}}{2(n+1)}\right|_0^1\right) = \lim_{n\to\infty}\left(0 + \frac{n^2}{2(n+1)}\right) = +\infty.$$

Notice that

$\lim_{n\to\infty} f_n(x) \to \lim_{n\to\infty}(\text{maximum}(f_n(x)_{x\in[0,1]})) \Rightarrow x \to 0$.

We can check this by taking the derivative:

Since $f_n(0) = f_n(1) = 0$, then f_n attains its maximum on $(0, 1)$.

$f_n'(x) = n^2(1-x^2)^n + n^2xn(1-x^2)^{n-1}(2x)$

$= n^2(1-x^2)^{n-1}(1 - x^2 - 2x^2n) = 0 \Rightarrow 1 - x^2 - 2x^2n = 0$ (as $x \neq 1$)

$$\Rightarrow x^2 = \frac{1}{1+2n} \Rightarrow x = \frac{1}{\sqrt{1+2n}} \Rightarrow \lim_{n\to\infty} x = \lim_{n\to\infty} \frac{1}{\sqrt{1+2n}} = 0$$

And $\lim_{n\to\infty} f_n\left(\frac{1}{\sqrt{1+2n}}\right)$

$$= \lim_{n\to\infty}\left[n^2 \frac{1}{\sqrt{1+2n}}\left(1-\frac{1}{1+2n}\right)^n\right] = \lim_{n\to\infty}\left[\frac{n^2}{\sqrt{1+2n}}\left(\frac{2n}{1+2n}\right)^n\right] = +\infty.$$

Example **7.5** Define $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

We will show $\lim_{n\to\infty} f_n'(x) \neq f'(x)$.

The limit function $f(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0$ since $-1 \leq \sin nx \leq 1$.

Thus, $f'(x) = 0$.

But $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \sqrt{n}\cos nx = \lim_{n\to\infty} \frac{n\cos nx}{\sqrt{n}}$ does not exist since

$f_n'(0) \to +\infty$, and $f_n'((2k+1)\pi) \to -\infty$, $k \in \mathbb{N}$.

Note Read through all the examples on pp. 144 - 146.

Example **7.2** Define $s_{m,n} = \dfrac{m}{m+n}$, $m, n \in \mathbb{N}$.

Then $\lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} = \lim_{n\to\infty}(1) = 1 \neq 0 = \lim_{m\to\infty}(0) = \lim_{m\to\infty}\lim_{n\to\infty} s_{m,n}$.

Example **7.3** Define $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Note that for each $n \in \mathbb{N}$, $f_n(x)$ is continuous, but its sum

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \begin{cases} 0 : x = 0 \\ 1 : x \neq 0 \end{cases}$$ is not continuous.

Example **7.4** Define $f_m(x) = \lim_{n\to\infty}(\cos m!\pi x)^{2n}$, $m, n \in \mathbb{N}$.

Then $\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n} = \begin{cases} 0 : x \text{ irrational} \\ 1 : x \text{ rational} \end{cases}$. This is an everywhere discontinuous limit function, which is not Riemann integrable.

Definition **7.7** Given $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$, $E \subset X$, a metric space, we say f_n *converges uniformly* if

$\forall\, \varepsilon > 0, \exists\, N \in \mathbb{N} \ni n \geq N \Rightarrow |f_n(x) - f(x)| \leq \varepsilon, \forall\, x \in E$

and *converges pointwise* if

$\forall\, x \in E, \forall\, \varepsilon > 0, \exists\, N \in \mathbb{N} \ni n \geq N \Rightarrow |f_n(x) - f(x)| \leq \varepsilon.$