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Theorem 7.9 $f_n \rightrightarrows f \Leftrightarrow M_n \to 0$

Theorem 7.13 If $K \subset X$ is compact, (a) $f_n : K \to \mathbb{R}$ is continuous $\forall\ n \in \mathbb{N}$, (b) $f_n \to f$ on K, f is continuous, and (c) $f_n(x) \geq f_{n+1}(x)\ \forall\ x \in K, \forall\ n \in \mathbb{N}$, then $f_n \rightrightarrows f$ on K.

Theorem 7.8 Cauchy Criterion
$f_n \rightrightarrows f$ on $E \Leftrightarrow \forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N} \ni \forall\ n, m \geq N, \forall\ x \in E, |f_n(x) - f_m(x)| < \varepsilon$.

Theorem 7.10 If $|f_n(x)| \leq M_n\ \forall\ x \in E, \forall\ n \in \mathbb{N}$, then ΣM_n converges $\Rightarrow \Sigma f_n$ converges uniformly.

Notation f_n converges pointwise to f is denoted by $f_n \to f$.
f_n converges uniformly to f is denoted by $f_n \rightrightarrows f$.

Theorem **7.9** $f_n \rightrightarrows f \Leftrightarrow M_n \to 0$ where
$\lim_{n\to\infty} f_n(x) = f(x)$ and $M_n = \sup_{x\in E}\left|f_n(x) - f(x)\right| < \varepsilon$ for each $n \in \mathbb{N}$.

Proof:
An immediate consequence of Definition 7.7 (uniform convergence).

Theorem **7.13** Suppose $K \subset X$ is compact, and
(a) $f_n : K \to \mathbb{R}$ is continuous $\forall\ n \in \mathbb{N}$,
(b) $\{f_n\}$ converges pointwise to a continuous function f on K,
(c) $f_n(x) \geq f_{n+1}(x)\ \forall\ x \in K, \forall\ n \in \mathbb{N}$, then
$f_n \rightrightarrows f$ on K.

Proof:
Let $g_n(x) = f_n(x) - f(x)$. Without loss of generality, suppose $f \equiv 0$.
Then we have g_n continuous $\forall\ n \in \mathbb{N}$, $g_n \to 0$ (pointwise), and
$g_n(x) \geq g_{n+1}(x)\ \forall\ x \in K, \forall\ n \in \mathbb{N}$.
Let $\varepsilon > 0$.
Let $K_n = \{x \in K: g_n(x) \geq \varepsilon\}$ for each $n \in \mathbb{N}$.
Each K_n is closed as $g_n^{-1}([\varepsilon, +\infty))$ is closed by continuity of g_n.
And K_n is bounded as $K_n \subset K$. $\therefore$ K_n is compact.
Since $g_n \geq g_{n+1}$, then $K_{n+1} \subset K_n$.
Since $g_n \to 0$, then $\forall\ x \in K, \exists\ N \in \mathbb{N} \ni g_n(x) < \varepsilon\ \forall\ n \geq N$, hence $x \notin K_N$.
$\therefore$ $\cap_{n\in\mathbb{N}} K_n = \varnothing$. So then $\exists\ N_1 \in \mathbb{N} \ni K_{N_1} = \varnothing$. $\therefore$ $0 \leq g_n(x) < \varepsilon\ \forall\ n \geq N_1$.
That is, $M_n = \sup_{x\in K}\left|g_n(x) - 0\right| < \varepsilon\ \forall\ n \geq N_1$.
$\therefore$ $\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N} \ni n \geq N \Rightarrow |\ g_n(x) - 0| \leq \varepsilon, \forall\ x \in K$, hence
$\forall\ \varepsilon > 0, \exists\ N \in \mathbb{N} \ni n \geq N \Rightarrow |\ f_n(x) - f(x)| \leq \varepsilon, \forall\ x \in K$.

Example Counterexample to previous theorem if K is not compact:

Define $f_n(x) = \dfrac{1}{nx+1}$ $(0 < x < 1, n \in \mathbb{N})$. Then $\lim_{n\to\infty} f_n(x) = f(x) = 0$.

Note that $f_n'(x) = \dfrac{-n}{(nx+1)^2}$ so $f(x)$ is decreasing on (0, 1).

And so we have f_n is continuous, $f_{n+1}(x) \le f_n(x), f_n(x) \to 0$, but

$M_n = \sup_{x\in(0,1)} |f_n(x) - f(x)| = \sup_{x\in(0,1)} |f_n(x)| = 1$.

In particular, for $\varepsilon = 1/2$, $\forall\, n \in \mathbb{N}$, $f_n(x) \ge 1/2$ for $x = 1/n$.

Theorem **7.8** Cauchy Criterion

$f_n \rightrightarrows f$ on $E \Leftrightarrow \forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} \ni \forall\, n, m \ge N, \forall\, x \in E, |f_n(x) - f_m(x)| < \varepsilon$.

Proof:

$\Rightarrow$: Assume $f_n \rightrightarrows f$ on E. Let $\varepsilon > 0$.

Then $\exists\, N \in \mathbb{N} \ni n \ge N$ and $x \in E \Rightarrow |f_n(x) - f(x)| < \varepsilon/2$.

Thus, for $n, m \ge N$, $|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f(x) - f_m(x)| \le \varepsilon$.

$\Leftarrow$: Assume $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} \ni \forall\, n, m \ge N, \forall\, x \in E, |f_n(x) - f_m(x)| < \varepsilon$.

Let $\varepsilon > 0$. Let $x_0 \in E$.

Then $|f_n(x_0) - f_m(x_0)| < \varepsilon \Rightarrow f_n(x_0) \to k$ where $k = f(x_0)$ for some function f.

Since this is true $\forall\, x \in E, f_n \to f$.

Fix $n \ge N$. Then $\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \le \varepsilon, \forall\, x \in E$.

Since this is true $\forall\, n \ge N$, $f_n \rightrightarrows f$ on E.

Remark Now we can consider series of functions:

$\sum_{n=1}^{\infty} f_n(x) = S(x)\ \forall\, x \in E$ if $S_n(x) \to S(x)\ \forall\, x \in E$ where

$S_n(x) = \sum_{k=1}^{n} f_k(x)$.

Theorem **7.10** If $|f_n(x)| \le M_n\ \forall\, x \in E, \forall\, n \in \mathbb{N}$, then

ΣM_n converges $\Rightarrow \Sigma f_n$ converges uniformly.

Proof:

Let $\varepsilon > 0$. ΣM_n converges $\Rightarrow \exists\, N \in \mathbb{N} \ni \forall\, n, m \ge N$, $\sum_{i=n}^{m} M_i < \varepsilon$. So then

$\left|\sum_{i=n}^{m} f_i(x)\right| \le \sum_{i=n}^{m} |f_i(x)|$

$= |f_n(x)| + |f_{n+1}(x)| + \cdots + |f_m(x)| \le M_n + M_{n+1} + \cdots + M_m = \sum_{i=n}^{m} M_i < \varepsilon$.

$\therefore\ M_n \to 0$, hence, by Theorem 7.9, Σf_n converges uniformly.