**Content**:

Theorem 7.11 Suppose $f_n \rightrightarrows f$ on $E \subset X$. Let $x \in E'$ and suppose $\lim_{t \to x} f_n(t) = A_n$ for $n = 1, 2, \ldots$, then $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

Theorem 7.12 Suppose $f_n \rightrightarrows f$ on $E \subset X$ where $\{f_n\}$ is a sequence of continuous functions, then $f$ is continuous on $E$.

Theorem 7.18 There exists a real continuous function on the real line which is nowhere differentiable.

**Theorem** **7.11** Suppose $f_n \rightrightarrows f$ on $E \subset X$. Let $x \in E'$ and suppose $\lim_{t \to x} f_n(t) = A_n$ for $n = 1, 2, \ldots$, then $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

**Proof**:

$f_n \rightrightarrows f \Rightarrow \forall\ \varepsilon > 0, \exists\ N_\varepsilon \ni |f_n(t) - f_m(t)| \le \varepsilon\ \forall\ n, m \ge N_\varepsilon, \forall\ t \in E$.

Then for $n, m \ge N_\varepsilon$, $\lim_{t \to x} |f_n(t) - f_m(t)| = |A_n - A_m| \le \varepsilon$.

$\therefore\ \{A_n\}$ is Cauchy, hence $A_n \to A$. Let $\varepsilon > 0$.

$f_n \rightrightarrows f \Rightarrow \exists\ N_1 \in \mathbb{N} \ni n \ge N_1 \Rightarrow |f(t) - f_n(t)| < \varepsilon/3\ \forall\ t \in E$.

$A_n \to A \Rightarrow \exists\ N_2 \in \mathbb{N} \ni n \ge N_2 \Rightarrow |A_n - A| < \varepsilon/3$.

Let $N = \max\{N_1, N_2\}$. Let $n \ge N$.

$x \in E' \Rightarrow \exists\ \delta > 0$ and $\exists\ t \in (t - \delta, t)$ or $t \in (t, t + \delta) \ni |f_n(t) - A_n| < \varepsilon/3$.

$\therefore\ |f(t) - A| \le |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Hence $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

**Theorem** **7.12** (Corollary to Thm 7.11)

Suppose $f_n \rightrightarrows f$ on $E \subset X$ where $\{f_n\}$ is a sequence of continuous functions, then $f$ is continuous on $E$.

**Proof**:

Let $\varepsilon > 0$. Let $x \in E$. Continuity of each $f_n$ give us that for each $n$,

$\exists\ \delta > 0 \ni |t - x| < \delta \Rightarrow |f_n(t) - f_n(x)| < \varepsilon/3$.

$f_n \rightrightarrows f \Rightarrow \exists\ N_\varepsilon \ni |f_n(t) - f(t)| \le \varepsilon/3\ \forall\ n \ge N_\varepsilon, \forall\ t \in E$.

So then replacing $A_n$ with $f_n(x)$ and $A$ with $f(x)$ in the proof above we have

$|f(t) - f(x)| \le |f(t) - f_n(t)| + |f_n(t) - f_n(x)| + |f_n(x) - f(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

**Theorem** **7.18** There exists a real continuous function on the real line which is nowhere differentiable.

**Proof**:
Define $\varphi(x) = |x|$ $(-1 \le x \le 1)$.
We can definie $\overline{\varphi}(x+2) = \overline{\varphi}(x)$ where $\overline{\varphi}\big|_{[-1,1]} = \varphi$.
For ease of notation we will simply call $\overline{\varphi}$, φ.
Define $f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \varphi(4^n x)$.
φ is continuous, and so, by Theorem 7.12, $f$ is continuous.
Let $\delta_m = \pm(1/2)4^{-m}$ where the sign is chosen so that no integer lies between $4^m x$ and $4^m(x + \delta_m)$.
Define $\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}$. Notice that for $n > m$,
$4^n\delta_m = \pm 4^n(1/2)4^{-m} = \pm(1/2)(2^2)^r = \pm 2^{2r-1}$ where $r > 0$.
Thus $\varphi(4^n(x + \delta_m)) = \varphi(4^n(x))$ by periodicity of φ, hence $\gamma_n = 0$.
For $n = m$, $4^n\delta_m = \pm 4^n(1/2)4^{-m} = \pm 1/2$, and for $n < m$, $|4^n\delta_m| < (1/2)4^{-m}$, hence
$\varphi(4^n(x + \delta_m)) - \varphi(4^n(x)) \le 4^n\,[|x + \delta_m| - |x|] = |4^n\delta_m| < (1/2)4^{-m}$.

So then $\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \frac{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \varphi(4^n x + \delta_m) - \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \varphi(4^n x)}{\delta_m} =$

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \frac{\left(\varphi(4^n x + \delta_m) - \varphi(4^n x)\right)}{\delta_m} =$$

$$\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \frac{\left(\varphi(4^n x + \delta_m) - \varphi(4^n x)\right)}{\delta_m} + \sum_{n=m+1}^{\infty}\left(\frac{3}{4}\right)^n \frac{\left(\varphi(4^n x + \delta_m) - \varphi(4^n x)\right)}{\delta_m} =$$

$$\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \frac{\left(\varphi(4^n x + \delta_m) - \varphi(4^n x)\right)}{\delta_m} + 0 =$$

$$\left(\frac{3}{4}\right)^m \frac{\left(\varphi(4^m x + \delta_m) - \varphi(4^m x)\right)}{\delta_m} + \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \frac{\left(\varphi(4^n x + \delta_m) - \varphi(4^n x)\right)}{\delta_m} =$$

$$\left(\frac{3}{4}\right)^m \frac{\pm 1/2}{(\pm 1/2)4^{-m}} + \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \frac{\left(\varphi(4^n x + \delta_m) - \varphi(4^n x)\right)}{\delta_m} \ge 3^m - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \frac{4^n\delta_m}{\delta_m} =$$

$$3^m - \sum_{n=0}^{m-1} 3^n = 3^m - 3^{m-1} - \cdots - 3^0 =$$

$$3^m - \frac{1-3^m}{1-3} = 3^m - \frac{3^m - 1}{3-1} = \frac{2 \bullet 3^m - 3^m + 1}{2} = \frac{3^m + 1}{2}.$$

$\therefore$ $f$ is nowhere differentiable.