Content:

Theorem 7.16 Consider α: $[a, b] \to \mathbb{R}$, bounded and monotonically increasing. Suppose that for f_n:$[a, b] \to \mathbb{R}$, $f_n \in \mathscr{R}(\alpha)$ $\forall$ $n \in \mathbb{N}$ and $f_n \rightrightarrows f$ on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = \int_a^b \lim_{n\to\infty}(f_n) d\alpha = \lim_{n\to\infty}\int_a^b f_n d\alpha$.

Corollary If $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ $\forall$ $n \in \mathbb{N}$ and converges uniformly on $[a, b]$, $S(x) = \Sigma_{n=1}^{\infty} f_n(x)$, then $S \in \mathscr{R}(\alpha)$ and $\int_a^b S d\alpha = \Sigma_{n=1}^{k}\int_a^b f_n d\alpha$.

Theorem **7.16** Consider α: $[a, b] \to \mathbb{R}$, bounded and monotonically increasing. Suppose that for f_n:$[a, b] \to \mathbb{R}$, $f_n \in \mathscr{R}(\alpha)$ $\forall$ $n \in \mathbb{N}$ and $f_n \rightrightarrows f$ on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = \int_a^b \lim_{n\to\infty}(f_n) d\alpha = \lim_{n\to\infty}\int_a^b f_n d\alpha$.

Proof:

Let $\varepsilon_n = \sup_{a\le x\le b} |f_n(x) - f(x)|$, then $f_n \rightrightarrows f \Rightarrow \varepsilon_n \to 0$.

And for each n, $|f_n(x) - f(x)| \le \varepsilon_n$, hence $f_n(x) - \varepsilon_n \le f(x) \le f_n(x) + \varepsilon_n$.

So then, $L(f_n(x) - \varepsilon_n, P, \alpha) \le L(f, P, \alpha) \le U(P, f, \alpha) \le U(f_n(x) - \varepsilon_n, P, \alpha)$.

Since $L(f_n(x) - \varepsilon_n, P, \alpha) = \sum_{i=1}^k (m_i - \varepsilon_n)\Delta\alpha_i = \sum_{i=1}^k m_i\Delta\alpha_i - \sum_{i=1}^k \varepsilon_n\Delta\alpha_i =$ $L(f_n, P, \alpha) - \varepsilon_n(\alpha(b) - \alpha(a))$ and, similarly,

$U(f_n(x) + \varepsilon_n, P, \alpha) = U(f_n, P, \alpha) + \varepsilon_n(\alpha(b) - \alpha(a))$.

Thus, $U(P, f, \alpha) - L(f, P, \alpha) \le U(f_n, P, \alpha) - L(f_n, P, \alpha) + 2\varepsilon_n(\alpha(b) - \alpha(a))$.

Let $\varepsilon > 0$. Then $\exists$ $N_\varepsilon \in \mathbb{N} \ni \varepsilon_n < \dfrac{\varepsilon}{4[\alpha(b) - \alpha(a)]}$ $\forall$ $n \ge N_\varepsilon$.

Let $n > N_\varepsilon$. Choose $P \ni U(f_n, P, \alpha) - L(f_n, P, \alpha) < \varepsilon/2$.

Then $U(P, f, \alpha) - L(f, P, \alpha) < \varepsilon$. $\therefore$ $f \in \mathscr{R}(\alpha)$.

By Theorem 6.12 (b) $\int_a^b (f_n(x) - \varepsilon_n) d\alpha \le \int_a^b f(x) d\alpha \le \int_a^b (f_n(x) + \varepsilon_n) d\alpha$

$\Rightarrow \int_a^b f_n(x) - \varepsilon_n(\alpha(b) - \alpha(a)) \le \int_a^b f d\alpha \le \int_a^b f_n(x) + \varepsilon_n(\alpha(b) - \alpha(a))$

$\Rightarrow \left|\int_a^b f d\alpha - \int_a^b f_n d\alpha\right| \le 2\varepsilon_n(\alpha(b) - \alpha(a))$.

Since $\lim_{n\to\infty}\varepsilon_n = 0$, then $\int_a^b f d\alpha = \lim_{n\to\infty}\int_a^b f_n d\alpha$.

Corollary If $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ $\forall$ $n \in \mathbb{N}$ and $S(x) = \Sigma_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$, then $S \in \mathscr{R}(\alpha)$ and $\int_a^b S d\alpha = \lim_{k\to\infty}\int_a^b S_k d\alpha = \Sigma_{n=1}^{k}\int_a^b f_n d\alpha$.

Proof:

Let $S_k(x) = \Sigma_{n=1}^{k} f_n(x)$, then $S_k(x) \rightrightarrows S(x)$. Then $S_k(x) \in \mathscr{R}(\alpha)$ by Theorem 6.12 as $\int_a^b S_k d\alpha = \sum_{n=1}^k \int_a^b f_n d\alpha$.

By Theorem 7.16, above, $S \in \mathscr{R}(\alpha)$ and $\int_a^b S d\alpha = \lim_{k\to\infty}\int_a^b S_k d\alpha = \sum_{n=1}^{\infty}\int_a^b f_n d\alpha$