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Definition 7.22 Equicontinuous functions

Theorem 7.24 If K is a compact metric space, $f_n \in \mathscr{C}(K)\ \forall\ n \in \mathbb{N}$, and $f_n \rightrightarrows f$ on K, then $\{f_n\}$ is equicontinuous on K.

Theorem 7.23 If E is countable and $\{f_n\}$ is pointwise bounded on E, then $\exists\ \{f_{n_k}\} \ni \{f_{n_k}\}$ converges $\forall\ x \in E$.

Theorem 7.25 If K is compact, $f_n \in \mathscr{C}(K)\ \forall\ n \in \mathbb{N}$, $\{f_n\}$ is pointwise bounded on K, and $\{f_n\}$ is equicontinuous on K, then
(a) $\{f_n\}$ is uniformly bounded on K,
(b) $\{f_n\}$ contains a uniformly convergent subsequence.

Definition **7.22** Let $\mathscr{F} \subset \{f : E \to \mathbb{C}\}$. We say $\mathscr{F}$ is equicontinuous if
$\forall\ \varepsilon > 0, \exists\ \delta > 0 \ni |x - y| < \delta$ and $x, y \in E \Rightarrow |f(x) - f(y)|\ \forall\ f \in \mathscr{F}$.

Theorem **7.24** If K is a compact metric space, $f_n \in \mathscr{C}(K)\ \forall\ n \in \mathbb{N}$, and $f_n \rightrightarrows f$ on K, then $\{f_n\}$ is equicontinuous on K.

Proof:
$f_n \rightrightarrows f \Rightarrow f$ is continuous by Theorem 7.12.
And compactness of K gives us that f is uniformly continuous.
Let $\varepsilon > 0$. Then $f_n \rightrightarrows f \Rightarrow \exists\ N_\varepsilon \in \mathbb{N} \ni \forall\ n > N_\varepsilon\ |f_n(x) - f(x)| < \varepsilon/3\ \forall\ n \in \mathbb{N}$.
By uniform continuity of f, $\exists\ \delta^* > 0 \ni d(x, y) < \delta^*$ and $x, y \in K \Rightarrow |f(x) - f(y)| < \varepsilon/3$.
Let $n > N_\varepsilon$ and let $x, y \in K \ni d(x, y) < \delta^*$.
Then $|f_n(x) - f_n(y)| \le |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| < \varepsilon$.
By uniform continuity of each f_n,
$\exists\ \delta_1, ..., \delta_{N_\varepsilon} > 0 \ni d(x, y) < \delta_i$ and $x, y \in K \Rightarrow |f_i(x) - f_i(y)| \le \varepsilon/3\ \forall\ i \in \{1, 2, ..., N_\varepsilon\}$.
Let $\delta = \min\{\delta_1, ..., \delta_{N_\varepsilon}, \delta^*\}$. Let $n > N_\varepsilon$.
Then, $|f_n(x) - f_n(y)| \le |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| < \varepsilon\ \forall\ n \in \mathbb{N}$.
Hence $\{f_n\}$ is equicontinuous on K.

Theorem **7.23** If E is countable and $\{f_n\}$ is pointwise bounded on E, then $\exists\ \{f_{n_k}\} \ni \{f_{n_k}\}$ converges $\forall\ x \in E$.

Proof:
Since E is countable, then we can let $E = \{x_1, x_2, ..., x_n\}$.
For x_1, $\{f_n(x_1)\}$ is bounded, so $\exists\ \{f_{1k}(x_1)\} \subset \{f_n(x_1)\}$ that converges.
For x_2, $\{f_{1k}(x_1)\}$ is bounded, so $\exists\ \{f_{2m}(x_2)\} \subset \{f_{1k}(x_1)\}$ that converges.
Note that $f_{11}, f_{12}, f_{13}, ...$ converge only for x_1;
$f_{21}, f_{22}, f_{23}, ...$ converge only for x_1 and x_2; and, in general,
$f_{n1}, f_{n2}, f_{n3}, ...$ converge only for $x_1, x_2, .., x_n$.
Select $f_{n1} = f_{11}, f_{n2} = f_{22}, f_{n3} = f_{33}, ...$. Then $\{f_{n_k}\}$ is a subsequence that converges at each $x \in E$.

Theorem **7.25** If K is compact, $f_n \in \mathscr{C}(K)$ $\forall\, n \in \mathbb{N}$, and $\{f_n\}$ is pointwise bounded on K, and $\{f_n\}$ is equicontinuous on K, then
(a) $\{f_n\}$ is uniformly bounded on K,
(b) $\{f_n\}$ contains a uniformly convergent subsequence.

Proof:
(a) Let $\varepsilon > 0$. Then equicontinuity of $\{f_n\} \Rightarrow$
$\exists\, \delta > 0 \ni d(x, y) < \delta$ and $x, y \in K \Rightarrow |f_n(x) - f_n(y)| < \varepsilon\ \forall\, n \in \mathbb{N}$.
By compactness of K, $\exists\ p_1, p_2, ..., p_r \in K \ni K \subset \cup_{i=1}^{r}\{B(p_i,\delta)\}$.
Pointwise boundedness of $\{f_n\} \Rightarrow \exists\, M_i > 0 \ni |f_n(p_i)| \le M_i\ \forall\, n \in \mathbb{N}, \forall\, i \in \{1, ..., r\}$.
Let $M = \max\{M_1, ..., M_r\}$. Let $x \in K$. Then $\exists\, i_0 \ni x \in B(p_{i_0}, \delta)$.
Thus, $|f_n(x)| \le |f_n(x) - f_n(p_{i_0})| + |f_n(p_{i_0})| \le \varepsilon + M\ \forall\, x \in K, \forall\, n \in \mathbb{N}$.

(b) Let E be a countable dense subset of K.
Equicontinuity of $\{f_n\} \Rightarrow$
$\exists\, \delta > 0 \ni d(x, y) < \delta$ and $x, y \in K \Rightarrow |f_n(x) - f_n(y)| < \varepsilon/3\ \forall\, n \in \mathbb{N}$.
By Theorem 7.23, and pointwise boundedness by part (a),
$\exists\, \{f_{n_k}\} \subset \{f_n\} \ni \{f_{n_k}\}$ converges $\forall\, x \in E$.
So $\exists\, N_\varepsilon \in \mathbb{N} \ni \forall\, n_k, n_m \ge N_\varepsilon$, we have $|f_{n_k}(x) - f_{n_m}(x)| \le \varepsilon/3\ \forall\, x \in E$.
By compactness of K, $\exists\ x_1, x_2, ..., x_r \in E \ni K \subset \cup_{i=1}^{r}\{B(x_i,\delta)\}$.
Let $x \in K$. Then $\exists\, i \ni x \in B(x_i, \delta)$. Thus, for $n_k, n_m \ge N_\varepsilon$.

$$|f_{n_k}(x) - f_{n_m}(x)| \le |f_{n_k}(x) - f_{n_k}(x_i)| + |f_{n_k}(x_i) - f_{n_m}(x_i)| + |f_{n_m}(x_i) - f_{n_m}(x)| \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$\therefore$ $\{f_{n_k}\}$ is uniformly convergent.