1. Write the definitions of the following objects.

(a) A metric space.

A pair (X, d) where X is a set and d: $X \times X \to \mathbb{R}$ is a function such that
$\forall\, x, y, z \in X$
(M1) $d(x, y) \geq 0$,
(M2) $d(x, y) = 0 \Leftrightarrow x = y$
(M3) $d(x, y) = d(y, x)$
(M4) $d(x, y) \leq d(x, z) + d(z, y)$.

(b) A normed space.

A vector space with a norm function, $||\bullet|| : X \to \mathbb{R}$, defined on it such that
$\forall\, x, y \in X, \forall\, \alpha \in K$,
(N1) $|| x || \geq 0$
(N2) $|| x || = 0 \Leftrightarrow x = 0$
(N3) $|| \alpha x|| = |\alpha|||x||$
(N4) $||x + y|| \leq || x || + || y ||$

(c) A Banach space.

A complete normed space.

(d) A Hamel basis of a vector space.

A linearly independent subset of a vector space, X, which spans X.

(e) A Schauder basis of a normed space.

A sequence $(e_n) \subset X$, a vector space, such that $\forall\, x \in X$, $\exists\, (a_n) \subset K$ and $n \in \mathbb{N}$ such that $|| x - \Sigma\alpha_n e_n || < \varepsilon$.

(f) $(l^p, ||\bullet||_p)$ Banach space.

A complete normed vector space in which $l^p = \{(x_n)_{n\in\mathbb{N}} | \, \Sigma|x_n|^p < +\infty\}$ and $\forall\, x \in X, || x ||_p = (\Sigma|x_n|^p)^{1/p}$ for $p \geq 1$.

(g) $(\mathscr{C}\,[a, b], ||\bullet||)$ Banach space.

A complete normed vector space in which $\mathscr{C}\,[a, b] = \{f :[a, b] \to \mathbb{R} \mid f$ is continuous$\}$ and
$\forall\, f \in \mathscr{C}\,[a, b], ||f|| = \max\limits_{x\in[a,b]} |f(x)|$.

(h) span (M) where M is a subset of a vector space X.

$\text{span}(M) = \{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_1, \alpha_2, ..., \alpha_n \in K, x_1, x_2, ..., x_n \in M\}$.

(i) The dimension of a Banach space.

The cardinality of the basis of the Banach vector space.

2. Give examples of the following facts. Explanations with proofs and (or) counterexamples are required.

(a) A normed space which is not a Banach space.

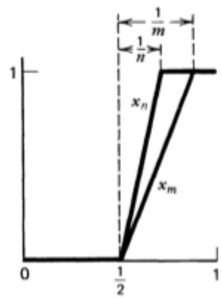
Let $X = \mathscr{C}[0, 1]$. Let $d: X \times X \to \mathbb{R}$ be defined by $d(f, g) = \int_0^1 |f(t) - g(t)dt|$.

Then (X, d) is not complete, hence not a Banach space.

Proof:

We will construct a Cauchy sequence of functions that limits to a function outside of X.

Define $f_m(x) = \begin{cases} 0 : 0 \le x \le \frac{1}{2} \\ \frac{1}{m}(x - \frac{1}{2}) : \frac{1}{2} < x < \frac{1}{2} + \frac{1}{m}. \\ 1 : \frac{1}{2} + \frac{1}{m} \le x \le 1 \end{cases}$



Let $\varepsilon > 0$. Let $N > 1/\varepsilon$. Then $\forall\ n > m > N$,

$d(f_n, f_m) = \frac{n-m}{2nm} < \frac{n}{2nm} = \frac{1}{2m} < \frac{1}{N} < \varepsilon$.

$\therefore\ f_m$ is Cauchy.

However, as $m \to +\infty$, $1/2 + 1/m \to 1/2$, so then $f_m(x) \to f(x) = \begin{cases} 0 : x \le 1/2 \\ 1 : x > 1/2 \end{cases}$

and $f(x) \notin \mathscr{C}[0, 1]$. $\therefore$ (X, d) is not complete.

(b) A metric on a vector space which cannot be obtained from a norm.

Let s = {all real sequences}.

Let $d: s \times s \to \mathbb{R}$ be defined by $d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}$. Then

d cannot be obtained from a norm.

Proof:

This metric fails the homogeneity property of the norm as

$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\alpha\xi_n|}{1 + |\alpha\xi_n|} \ne |\alpha| \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n|}{1 + |\xi_n|}$.

2. (c) A Schauder basis which is not a Hamel basis.

Let X = l^p.

Let $e_n = (\delta_{nj})$ where

$\delta_{1j} = (1, 0, 0, \ldots)$, $\delta_{2j} = (0, 1, 0, 0, \ldots)$, ..., $\delta_{nj} = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$, ...

Then $B = \{e_1, e_2, \ldots\}$ is a Schauder basis and is not a Hamel basis.

Proof:

To show B is a Schauder basis, let $\mathbf{x} \in X$, and note that

$\mathbf{x} = (\alpha_1, \alpha_2, \ldots) = \Sigma\alpha_n(\delta_{nj}) = \Sigma\, \alpha_n e_n$ where $\alpha_1\ \alpha_2, \ldots \in K$.

$\therefore$ (δ_{nj}) is a Schauder basis for l^p.

To show B is not a Hamel basis, let $\mathbf{x} = (1/n^2)_{n\in\mathbb{N}}$.

Then $\forall\, n \in \mathbb{N}$ $\sum_{i=n}^{\infty} \frac{1}{i^2}\delta_{ij} = (0, 0, \ldots, \frac{1}{n^2}, \frac{1}{(n+1)^2}, 0, 0, \ldots) \neq \Theta$.

So then $\mathbf{x} - \sum_{i=n}^{\infty} \frac{1}{i^2}\delta_{ij} = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{(n-1)^2}, 0, 0, \ldots\right) \neq \mathbf{x}$.

$\therefore\ \nexists\, n \in \mathbb{N} \ni \mathbf{x} = \sum_{i=1}^{n} \frac{1}{i^2}(\delta_{ij})$, a finite sum of basis elements.

(d) An absolutely convergent series which is not convergent.

Let $Y \subset l^\infty$ such that Y is the set of all sequences with only finitely many terms. Let $e_n = (\delta_{nj})$ as defined in (c) above. Then for $\mathbf{x}_n = \frac{e_n}{n^2}$, $\Sigma\mathbf{x}_n \notin Y$.

Proof:

$\mathbf{x}_1 = (1, 0, 0, \ldots)$

$\mathbf{x}_2 = (0, 1/4, 0, 0, \ldots)$

$\mathbf{x}_n = (0, 0, \ldots 0, 1/n^2, 0, 0, \ldots)$

So for each n, $\mathbf{x}_n$ has only one non-zero term.

Hence, $(\mathbf{x}_n) \subset Y$.

And $\Sigma\|\mathbf{x}_n\| = \Sigma\sup|1/n^2|$ converges as it is a convergent p-series.

However, $\Sigma\mathbf{x}_n = (1, 1/4, 1/9, \ldots, 1/n^2, 1/(n+1)^2, \ldots)$, a sequence of infinitely many non-zero terms, hence not an element of Y.

(e) A Cauchy sequence which is not convergent.

Let $X = \mathbb{Z}$ and $d(m, n) = |m^{-1} - n^{-1}|$. Let $x_n = n$.

Then (x_n) is a non-convergent Cauchy sequence.

Proof:

Let $\varepsilon > 0$. Choose $N \in \mathbb{N} \ni N > 1/\varepsilon$.

Then $\forall\, n \geq N$, $|x_n - x_{n+1}| = |1/n - 1/(n+1)| < 1/n < \varepsilon$.

Thus (x_n) is Cauchy. However, (x_n) does not converge.

For if $\exists\, x \in \mathbb{R} \ni x_n \to x$, then x must be the least upper bound of (x_n) as x_n is increasing. However, we can find $n \in \mathbb{N}$ such that $n = x_n > x$, a contradiction.

3. Answer the following questions. Explanations with proofs and (or) counterexamples are required.

(a) Is $Y = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - 3x_3 = 0\}$ a subspace of $\mathbb{R}^3$?

Yes.

Proof:
Since $0 + 2\bullet 0 - 3\bullet 0 = 0$, then $(0, 0, 0) \in Y$, hence $Y \neq \varnothing$.
Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and let $\alpha \in K$.
Then $\alpha x + y = (\alpha x_1 + y_1, \alpha x_2 + y_2\ \alpha x_3 + y_3)$ and
$\alpha x_1 + y_1 + 2(\alpha x_2 + y_2) - 3(\alpha x_3 + y_3)$
$\alpha(x_1 + 2x_2 - 3x_3) + y_1 + 2y_2 - 3y_3 =$
$\alpha \bullet 0 + 0 = 0.$
$\therefore\ \alpha x + y \in Y.$
So then Y is a subspace of $\mathbb{R}^3$.

(b) Are $x_1(t) = t$ and $x_2(t) = t^2$ linearly independent in $\mathscr{C}\,[0, 1]$?

Yes.

Proof:
Suppose $\alpha_1 t + \alpha_2 t^{\,2} = 0$. If $\alpha_1 \neq 0$, then $t(\alpha_2 t + \alpha_1) = 0$, hence $t = 0$ or $t = -\alpha_1/\alpha_2$.
But this equation must be true for all $t \in [0, 1]$.
Thus, $\alpha_2 = 0$, which gives us that $\alpha_1 t = 0\ \forall\ t \in [0, 1]$, hence $\alpha_1 = 0$.
$\therefore\ x_1$ and x_2 are linearly independent in $\mathscr{C}\,[0, 1]$.

(c) In a Banach space does $\| x_n \| \to \| x \| \Rightarrow x_n \to x$?

No.

Proof:
Let $X = \{ z \in \mathbb{C} \mid \| z \| = 1\}$. Let $x_n = \cos \pi n + i \sin \pi n$.
Then $\| x_n \| = 1$ for all $n \in \mathbb{N}$, or, equivalently, $\| x_n \| \to 1$.
But x_n converges alternately to -1 and 1, hence, does not converge.

What about $x_n \to x$ implying $\| x_n \| \to \| x \|$?

Yes.

Proof:
Let $\varepsilon > 0$. If $x_n \to x$, then $\exists\ N \in \mathbb{N} \ni \| x_n - x \| < \varepsilon$.
$\| x_n \| = \| x_n - x + x \| \leq \| x_n - x \| + \| x \|$.
$\| x \| = \| x - x_n + x_n \| \leq \| x - x_n \| + \| x_n \|$.
Thus, $\big|\, \| x_n \| - \| x \| \,\big| \leq \| x_n - x \| < \varepsilon$. $\therefore\ \| x_n \| \to \| x \|$.