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6. Show that d in 1.1-6 (i.e. $d(x, y) = \sup_{j\in N}\left|\xi_j - \eta_j\right|$ where $x = (\xi_1,\xi_2,\ldots)$ and $y = (\eta_1,\eta_2,\ldots)$ for $j = 1, 2, \ldots$, $\left|\xi_j\right| \le c_x$ for some real number c_x, and $x, y \in X$ = {bounded sequences of complex numbers}) satisfies the triangle inequality.

Proof:
Let $x = (\xi_1,\xi_2,\ldots)$, $y = (\eta_1,\eta_2,\ldots)$, and $z = (\zeta_1,\zeta_2,\ldots)$. Then for all j,
$\left|\xi_j - \eta_j\right| = \left|\xi_j - \zeta_j\right| + \left|\zeta_j - \eta_j\right| \le \sup_{j\in N}\left|\xi_j - \zeta_j\right| + \sup_{j\in N}\left|\zeta_j - \eta_j\right| < 2c_{x-z} + 2c_{z-y} < +\infty$.
Since this is true for all j, we can take the supremum of the left side to get
$d(x, y) = \sup_{j\in N}\left|\xi_j - \eta_j\right| \le \sup_{j\in N}\left|\xi_j - \zeta_j\right| + \sup_{j\in N}\left|\zeta_j - \eta_j\right| = d(x, z) + d(z, y)$.

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7. If A is the subspace of l^∞ consisting of all sequences of zeros and ones, what is the induced metric on A?
If the metric defined on l^∞ is the same as defined in #6 above, then the induced metric on A is $d(x,y) = \begin{cases} 1 : x \ne y \\ 0 : x = y \end{cases}$.

Proof:
$\forall\, x, y \in A, \forall\, j \in \mathbb{N}$, $\left|\xi_j - \eta_j\right| = \begin{cases} 1 : \xi_j \ne \eta_j \\ 0 : \xi_j = \eta_j \end{cases}$, so then $d(x,y) = \begin{cases} 1 : x \ne y \\ 0 : x = y \end{cases}$.
M1, M2, M3 are evident.
To show the triangle inequality, M4, is satisfied, note that for $x = y$,
clearly we have $0 = d(x, y) \le d(x, z) + d(z, y)$.
For $x \ne y$, we have either $x \ne z$ or $y \ne z$, hence $d(x, z) = 1$ or $d(y, z) = 1$.
Thus, $1 = d(x, y) \le d(x, z) + 1$ or $1 = d(x, y) \le 1 + d(z, y)$.

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8. Show that another metric $\tilde{d}$ on the set X in 1.1-7 ($X = \mathscr{C}\,[a, b]$) is defined by
$\tilde{d}(x,y) = \int_a^b \left|x(t) - y(t)\right| dt$.

Proof:
M1, M3, and $\tilde{d}(x,x) = 0$ are evident. If $\tilde{d}(x,y) = 0$ for some $x, y \in X$, then $\left|x(t) - y(t)\right| = 0$ for all t, hence $x = y$. So M2 holds.
To show the triangle inequality, M4, is satisfied, let $x, y, z \in X$ and note that
$\int_a^b \left|x(t) - y(t)\right| dt = \int_a^b \left|x(t) - z(t) + z(t) - y(t)\right| dt \le \int_a^b \left(\left|x(t) - z(t)\right| + \left|z(t) - y(t)\right|\right) dt =$
$\int_a^b \left|x(t) - z(t)\right| dt + \int_a^b \left|z(t) - y(t)\right| dt$.
Since $|x(t) - z(t)|$ and $|z(t) - y(t)|$ are compositions of continuous functions, then we have $\tilde{d}(x,y) = \tilde{d}(x,z) + \tilde{d}(z,y)$.

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13. Using the triangle inequality, show that $|d(x, z) - d(y, z)| \leq d(x, y)$.
Proof:
By the triangle inquality, $d(x, z) \leq d(x, y) + d(y, z)$ and $d(y, z) \leq d(y, x) + d(x, z)$.
Thus $-d(x, y) \leq d(x, z) - d(y, z)$ and $d(x, z) - d(y, z) \leq d(x, y)$.
So then $|d(x, z) - d(y, z)| \leq d(x, y)$.