Section 1.2, page 16 # 2, 3, 4, 5

2. Using (6) $\alpha\beta \le \int_0^\alpha t^{p-1}dt + \int_0^\beta u^{q-1}du = \frac{\alpha^p}{p} + \frac{\beta^q}{q}$, show that the geometric mean of two positive numbers does not exceed the arithmetic mean.

Proof:

Let a, $b \in \mathbb{R}^+$. The geometric mean of a and b is $\sqrt{ab}$.

By (6), we have $\sqrt{ab} \le \frac{\sqrt{a}^2}{2} + \frac{\sqrt{b}^2}{2} = \frac{a+b}{2}$, the arithmetic mean of a and b.

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3. Show that the Cauchy-Schwarz inequality (11) $\sum_{j=1}^{\infty}|\xi_j\eta_j| \le \sqrt{\sum_{k=1}^{\infty}|\xi_k|^2}\sqrt{\sum_{m=1}^{\infty}|\eta_m|^2}$ implies $(|\xi_1|+\cdots+|\xi_n|)^2 \le n(|\xi_1|^2+\cdots+|\xi_n|^2)$.

Proof:

Let $x = (\xi_1,\xi_2,\ldots,\xi_n,0,0,\ldots)$ and $y = (\eta_1,\eta_2,\ldots,\eta_n,0,0,\ldots)$ where for each i, $1 \le i \le n$, $\eta_i = 1$.

Then by (11), we have $\sum_{j=1}^{\infty}|\xi_j| = \sum_{j=1}^{\infty}|\xi_j\eta_j| \le \sqrt{\sum_{k=1}^{\infty}|\xi_k|^2}\sqrt{\sum_{m=1}^{\infty}|\eta_m|^2} = n\sqrt{\sum_{k=1}^{\infty}|\xi_k|^2}$.

Thus $(|\xi_1|+\cdots+|\xi_n|+0+0+\cdots)^2 = \left(\sum_{j=1}^{\infty}|\xi_j|\right)^2 \le n\sum_{k=1}^{\infty}|\xi_k|^2 = n(|\xi_1|^2+\cdots+|\xi_n|^2+0+0+\cdots)$

as desired.

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4. Find a sequence which converges to 0, but is not in any space l^p, where $1 \le p < +\infty$.

Sequence: Let $x_n = \frac{1}{\ln n}$

Proof:

Let $p \ge 1$. Clearly $\frac{1}{\ln n} \to 0$. We want to show $\lim_{n\to\infty}\frac{\frac{1}{(\ln n)^p}}{\frac{1}{n}} = \infty$,

$$\lim_{n\to\infty}\frac{n}{(\ln n)^p} = \lim_{n\to\infty}\frac{1}{pn^{-1}(\ln n)^{p-1}} = \cdots = \lim_{n\to\infty}\frac{1}{p!n^{-1}(\ln n)} = \infty.$$

By the limit comparison test, $\sum_{n=2}^{\infty}\left(\frac{1}{\ln n}\right)^p$ diverges.

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5. Find a sequence, x, which is in l^p, with $p > 1$, but $x \notin l^1$.
Sequence: Let $x_n = 1/n$.
Proof:
$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but for $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a convergent p-series.